Elliptic Calogero-Moser Systems and Isomonodromic Deformations

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Abstract

We show that various models of the elliptic Calogero-Moser systems are accompanied with an isomonodromic system on a torus. The isomonodromic partner is a non-autonomous Hamiltonian system defined by the same Hamiltonian. The role of the time variable is played by the modulus of the base torus. A suitably chosen Lax pair (with an elliptic spectral parameter) of the elliptic Calogero-Moser system turns out to give a Lax representation of the non-autonomous system as well. This Lax representation ensures that the non-autonomous system describes isomonodromic deformations of a linear ordinary differential equation on the torus on which the spectral parameter of the Lax pair is defined. A particularly interesting example is the "extended twisted BC_ℓ model" recently introduced along with some other models by Bordner and Sasaki, who remarked that this system is equivalent to Inozemtsev's generalized elliptic Calogero-Moser system. We use the "root type" Lax pair developed by Bordner et al. to formulate the associated isomonodromic system on the torus.

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1 Introduction

In 1996, Manin [1] proposed a new expression of the sixth Painlevé equation. This is a differential equation of the form

$$(2\pi i)^2 \frac{d^2 q}{d\tau^2} = \sum_{a=0}^{3} \alpha_a \wp'(q + \omega_a), \tag{1.1}$$

where $\wp'(u)$ is the derivative of the Weierstrass $\wp$ function with primitive periods 1 and τ,

$$\wp(u) = \wp(u \mid 1, \tau) = \frac{1}{u^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(u + m + n\tau)^2} - \frac{1}{(m + n\tau)^2} \right), \tag{1.2}$$

ω_a $(a = 0, 1, 2, 3)$ are the origin and the three half-periods of the torus $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$,

$$\omega_0 = 0, \quad \omega_1 = \frac{1}{2}, \quad \omega_2 = \frac{1}{2} + \frac{\tau}{2}, \quad \omega_3 = \frac{\tau}{2}, \tag{1.3}$$

and α_a $(a = 0, 1, 2, 3)$ are the simple linear combinations $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha, -\beta, \gamma, 1/2 - \delta)$ of the four parameters α, β, γ and β of the sixth Painlevé equation

$$\begin{aligned} \frac{dy^2}{dx^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) \left(\frac{dy}{dx} \right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) \frac{dy}{dx} \\ &\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left(\alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right). \end{aligned} \tag{1.4}$$

Manin's equation can be written in the Hamiltonian form

$$2\pi i \frac{dq}{d\tau} = p, \quad 2\pi i \frac{dp}{d\tau} = -\frac{\partial \mathcal{H}}{\partial q} \tag{1.5}$$

with the Hamiltonian

$$\mathcal{H} = \frac{1}{2} p^2 - \sum_{a=0}^{3} \alpha_a \wp(q + \omega_a). \tag{1.6}$$

Since the Hamiltonian depends on the modulus τ explicitly, this is a non-autonomous Hamiltonian system. In this new framework, Manin reconsidered the affine Weyl group symmetries of the sixth Painlevé equation discovered by Okamoto [2], solutions for special values of α, β, γ and δ constructed by Hitchin [3], etc.

Manin's equation reveals an unexpected link between the Painlevé equation and the elliptic Calogero-Moser systems, i.e., the Calogero-Moser systems [4] with elliptic potentials. In order to see this relation, we introduce a new variable t and formally replace

$2\pi i d/d\tau \rightarrow d/dt$ in the aforementioned equations. The outcome are the autonomous equation

$$\frac{d^2q}{dt^2} = \sum_{a=0}^{3} \alpha_a \wp'(q + \omega_a) \tag{1.7}$$

and its Hamiltonian form

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}. \tag{1.8}$$

If all α_n's take the same value $-g^2/8$, one can use an identity of the $\wp$ function to rewrite the above equation as:

$$\frac{d^2q}{dt^2} = -\frac{g^2}{8} \sum_{a=0}^{3} \wp'(q + \omega_a) = -g^2 \wp'(2q). \tag{1.9}$$

This is exactly the two-body elliptic Calogero-Moser system; the ℓ-body elliptic Calogero-Moser system ($A_{\ell-1}$ model) is defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{j=1}^{\ell} p_j^2 + \frac{g^2}{2} \sum_{j \neq k} \wp(q_j - q_k). \tag{1.10}$$

As Krichever [5] demonstrated, this elliptic Calogero-Moser system is an isospectral integrable system with a Lax representation

$$\frac{\partial L(z)}{\partial t} = [L(z), M(z)], \tag{1.11}$$

where the Lax pair $L(z)$ and $M(z)$ are matrix-valued functions of a spectral parameter z on the torus E_τ. Furthermore, the general case falls into Inozemtsev's generalization of the elliptic Calogero-Moser system [6] defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{j=1}^{\ell} p_j^2 + \frac{g_m^2}{2} \sum_{\epsilon, \epsilon' = \pm 1} \sum_{j \neq k} \wp(\epsilon q_j + \epsilon' q_k) + \frac{1}{2} \sum_{j=1}^{\ell} \sum_{a=0}^{3} g_a^2 \wp(q_j + \omega_a). \tag{1.12}$$

Levin and Olshanetsky [7] developed a geometric formulation of isomonodromic systems on a general Riemann surface, and characterized Manin's equation as an isomonodromic system on the torus E_τ. Their interpretation of isomonodromic deformations is based on the notion of the Hitchin systems [8]. According to this interpretation, the coordinates q_j of Calogero-Moser particles are identified with the moduli of an $SU(\ell)$ flat

bundle on the torus E_τ, and the L-matrix $L(z)$ is nothing but the Higgs field on this bundle. (Such a link between the elliptic Calogero-Moser systems and the Hitchin systems was already pointed out before their work by Nekrasov [9] and Enriquez and Rubtsov [10].) Isomonodromic deformations are special deformations of these geometric data as the complex structure of the base torus (or, equivalently, the modulus τ) varies. This geometric picture suggests a wide range of generalizations of isomonodromic deformations (see, e.g., the recent work of Levin and Olshanetsky [11]).

Unfortunately, however, it is only the special case with $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$ that was successfully treated in the formulation of Levin and Olshanetsky. This is simply because no suitable Lax representation was available for the Inozemtsev system. Inozemtsev [6] presented a Lax representation, but it is not suited for that purpose.

Recently, a new type of Lax pair — the root type Lax pair — was proposed by Bordner et al. [12, 13, 14] for various models of the elliptic Calogero-Moser systems including the Inozemtsev system. This is a Lax pair constructed on the basis of an underlying root system (e.g., the $A_{\ell-1}$ root system for the aforementioned elliptic Calogero-Moser system, and the BC_ℓ root system for the Inozemtsev system). The construction covers not only the ordinary elliptic Calogero-Moser systems (the "untwisted models") but also the "twisted models" introduced by D'Hoker and Phong [15] and their generalizations (the "extended twisted models"). The Inozemtsev system coincides with the extended twisted BC_ℓ model in the classification of Bordner and Sasaki [14]. In particular, the root type Lax pair for the extended twisted BC_1 model gives a Lax representation to the aforementioned isospectral analogue of Manin's equation.

One of the goals of this paper is to show, using the root type Lax pair, that each of these elliptic Calogero-Moser systems are accompanied with an isomonodromic system on a torus. The fist step of the construction is simply to replace the equations of motions

$$\frac{dq}{dt} = \{q, \mathcal{H}\}, \quad \frac{dp}{dt} = \{p, \mathcal{H}\} \tag{1.13}$$

of the elliptic Calogero-Moser system by the non-autonomous system

$$2\pi i \frac{dq}{d\tau} = \{q, \mathcal{H}\}, \quad 2\pi i \frac{dp}{d\tau} = \{p, \mathcal{H}\} \tag{1.14}$$

with the same Hamiltonian $\mathcal{H}$. We then rewrite this non-autonomous system into a Lax

equation of the form

$$2\pi i\frac{\partial L(z)}{\partial \tau}+\frac{\partial M(z)}{\partial z}=[L(z),M(z)] \tag{1.15}$$

using a root type Lax pair $L(z)$ and $M(z)$. This Lax equation implies the Frobenius integrability of the linear system

$$\frac{\partial Y(z)}{\partial z}=L(z)Y(z), \quad 2\pi i\frac{\partial L(z)}{\partial \tau}+M(z)Y(z)=0, \tag{1.16}$$

from which one can deduce that the non-autonomous system is an isomonodromic system on the torus E_τ.

Actually, we shall use the root type Lax pair made of slightly different building blocks. The root type Lax pairs, like the previously known Lax pairs, contain complex analytic functions $x(u,z)$, $y(u,z)$, etc. that satisfy special functional equations (called the "Calogero functional equations" [16]). Bordner et al. use the Weierstrass sigma function to construct those functions. We use the Jacobi theta function θ_1 instead. This is inspired by the work of Levin and Olshanetsky, who used substantially the same function to construct the L-matrix (i.e., the Higgs field in their framework) for isomonodromic systems on a torus. This minuscule difference is rather crucial for deriving an isomonodromic Lax equation as above.

The functions $x(u,z)$ and $y(u,z)$ that we use are, in fact, identical to the functions that Felder and Wieczerkowski [17] used in their study on the Knizhnik-Zamolodchikov-Bernard (KZB) equation [18]. This is by no means a coincidence. As Levin and Olshanetsky stressed, the KZB equation and the Hitchin system (or, rather, its isomonodromic version) are closely related.

In order to illustrate that our method also works for some other cases, we show a construction of an isomonodromic analogue for the "spin generalization" [19] of the elliptic Calogero-Moser system. Actually, a multi-spin generalization of this construction is also possible, which is nothing but the genus-one case of Levin and Olshanetsky's framework.

This paper is organized as follows. In Section 2, we illustrate our construction of isomonodromic systems in the case of the most classical $A_{\ell-1}$ model. This will serve as a prototype of the subsequent discussion. Section 3 is devoted to the models treated by the root type Lax pairs, and Section 4 to the spin generalization. Section 5 is for concluding remarks. Technically complicated calculations are collected in Appendices.

2 Isomonodromic Systems on the Torus — a Prototype

We start with illustrating our construction for the most fundamental case — the the $A_{\ell-1}$ model and its Lax pair in the vector representation of $SU(\ell)$.

2.1 $A_{\ell-1}$ Model of Elliptic Calogero-Moser Systems

The $A_{\ell-1}$ model is defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{\ell} p_j^2 + \frac{g^2}{2}\sum_{j\neq k}\wp(q_j - q_k). \tag{2.1}$$

Here q_j and p_j $(j = 1, \cdots, \ell)$ are the coordinates and momenta of the particles with the canonical Poisson brackets

$$\{q_j, p_k\} = \delta_{jk}, \quad \{q_j, q_k\} = \{p_j, p_k\} = 0. \tag{2.2}$$

Following Manin's equation, we noralize the primitive periods as

$$2\omega_1 = 1, \quad 2\omega_3 = \tau \tag{2.3}$$

The equations of motion are give by the canonical equations

$$\begin{aligned} \frac{dq_j}{dt} &= \{q_j, \mathcal{H}\} = p_j, \\ \frac{dp_j}{dt} &= \{p_j, \mathcal{H}\} = -g^2\sum_{k\neq j}\wp'(q_j - q_k). \end{aligned} \tag{2.4}$$

This elliptic Calogero-Moser system has a Lax pair of the form

$$\begin{aligned} L(z) &= \sum_{j=1}^{\ell} p_j E_{jj} + ig\sum_{j\neq k} x(q_j - q_k, z)E_{jk}, \\ M(z) &= \sum_{j=1}^{\ell} D_j E_{jj} + ig\sum_{j\neq k} y(q_j - q_k, z)E_{jk}, \end{aligned} \tag{2.5}$$

where E_{jk} is the matrix unit, $(E_{jk})_{mn} = \delta_{mj}\delta_{nk}$. The diagonal elements D_j of $M(z)$ are given by

$$D_j = ig\sum_{k\neq j}\wp(q_j - q_k), \tag{2.6}$$

and $x(u,z)$ is a function that satisfies, along with its u-derivative

$$y(u,z) = \frac{\partial x(u,z)}{\partial u}, \tag{2.7}$$

the functional equations

$$x(u,z)y(v,z) - y(u,z)x(v,z) = x(u+v,z)(\wp(u) - \wp(v)), \tag{2.8}$$

$$x(u,z)y(-u,z) - y(u,z)x(-u,z) = \wp'(u), \tag{2.9}$$

$$x(u,z)x(-u,z) = \wp(z) - \wp(u). \tag{2.10}$$

Using these functional equations, one can easily prove the following well known result [5]:

Proposition 1 *The matrices $L(z)$ and $M(z)$ satisfy the Lax equation*

$$\frac{\partial L(u)}{\partial t} = [L(z), M(z)]. \tag{2.11}$$

As far as the elliptic Calogero-Moser system is concerned, the choice of $x(u,z)$ and $y(u,y)$ is rather irrelevant. A standard choice is the function

$$x(u,z) = \frac{\sigma(z-u)}{\sigma(z)\sigma(u)}, \tag{2.12}$$

where $\sigma(u) = \sigma(u \mid 1, \tau)$ is the Weierstrass sigma function with primitive periods 1 and τ.

Thus, the elliptic Calogero-Moser system is an isospectral integrable system. An involutive set of conserved quantities can be extracted from the traces $\operatorname{Tr} L(z)^k$, $k = 2, 3, \cdots$ of powers of the L-matrix. The quadratic trace is substantially the Hamiltonian itself:

$$\operatorname{Tr} \frac{L(z)^2}{2} = \mathcal{H} + (\text{independent of } p \text{ and } q). \tag{2.13}$$

The functions $x(u,z)$ and $y(u,z)$ based on the sigma function, however, are not very suited for constructing an isomonodromic system. We shall show an alternative in the next subsection.

2.2 Our choice of $x(u,z)$ and $y(u,z)$

Inspired by the work of Levin and Olshanetsky [7], we take the following function $x(u,z)$ and its u-derivative $y(u,z)$ for constructing an isomonodromic Lax pair:

$$x(u,z) = \frac{\theta_1(z-u)\theta_1'(0)}{\theta_1(z)\theta_1(u)}. \tag{2.14}$$

Here $\theta_1(u)$ is one of Jacobi's elliptic theta functions,

$$\theta_1(u) = \theta_1(u \mid \tau) = -\sum_{n=-\infty}^{\infty} \exp\left(\pi i \tau \left(n+\frac{1}{2}\right)^2 + 2\pi i \left(n+\frac{1}{2}\right)\left(u+\frac{1}{2}\right)\right), \tag{2.15}$$

and $\theta_1'(u)$ its derivative. Accordingly, the partner $y(u,z)$ can be written

$$y(u,z) = -x(u,z)(\rho(u)+\rho(z-u)), \tag{2.16}$$

where $\rho(u)$ denotes the logarithmic derivative of $\theta_1(u)$,

$$\rho(u) = \frac{\theta_1'(u)}{\theta_1(u)}. \tag{2.17}$$

The function $\rho(u)$, too, plays an important role throughout this paper.

Proposition 2 *These functions $x(u,z)$ and $y(u,z)$ satisfy the functional equations (2.8) – (2.10) and the differential equation*

$$2\pi i \frac{\partial x(u,z)}{\partial \tau} + \frac{\partial^2 x(u,z)}{\partial u \partial z} = 0. \tag{2.18}$$

The last differential equation (a kind of $1+2$-dimensional "heat equation") is a characteristic of our (x,y) pair, and plays a key role in our construction of isomonodromic systems.

We give a proof of these properties in Appendix A. The following are supplementary remarks on these functions.

- The proof of (2.8–2.10) is based on the following analytical properties of $x(u,z)$:

    1. $x(u,z)$ is a meromorphic function of u and z. The poles on the u plane and the z plane are both located at the lattice points $u = m + n\tau$ and $z = m + n\tau$ $(m, n \in \mathbb{Z})$.

2. $x(u,z)$ has the following quasi-periodicity:

$$\begin{aligned} x(u+1,z) &= x(u,z), \qquad x(u+\tau,z) = e^{2\pi iz}x(u,z), \\ x(u,z+1) &= x(u,z), \qquad x(u,z+\tau) = e^{2\pi iu}x(u,z). \end{aligned} \tag{2.19}$$

3. At the origin of the u and z planes, $x(u,z)$ exhibits the following singular behavior:

$$\begin{aligned} x(u,z) &= \frac{1}{u} - \rho(z) + O(u) \quad (u \to 0), \\ x(u,z) &= -\frac{1}{z} + \rho(u) + O(z) \quad (z \to 0). \end{aligned} \tag{2.20}$$

- These properties are an immediate consequence of the following well known fact:

1. $\theta_1(u)$ is an entire function with simple zeros at the lattice points $u = m + n\tau$ $(m, n \in \mathbb{Z})$.

2. $\theta_1(u)$ is an odd and quasi-periodic function,

$$\begin{aligned} \theta_1(-u) &= \theta_1(u+1) = -\theta_1(u), \\ \theta_1(u+\tau) &= -e^{-\pi i\tau - 2\pi iu}\theta_1(u). \end{aligned} \tag{2.21}$$

- One can similarly see the following analytical properties of $\rho(u)$:

1. $\rho(u)$ is a meromorphic function with poles at the lattice points $u = m + n\tau$ $(m, n \in \mathbb{Z})$.

2. $\rho(u)$ is an odd function with additive quasi-periodicity:

$$\rho(-u) = -\rho(u), \quad \rho(u+1) = \rho(u), \quad \rho(u+\tau) = \rho(u) - 2\pi i. \tag{2.22}$$

3. At the origin $u = 0$, $\rho(u)$ exhibits the following singular behavior:

$$\rho(u) = \frac{1}{u} + \frac{\theta_1'''(0)}{3\theta_1'(0)}u + O(u^3) \quad (u \to 0). \tag{2.23}$$

- The proof of (2.18) is based on the well known "heat equation"

$$4\pi i \frac{\partial \theta_1(u)}{\partial \tau} = \frac{\partial^2 \theta_1(u)}{\partial u^2}. \tag{2.24}$$

of the Jacobi theta function.

2.3 Isomonodromic deformations

Replacing $d/dt \to 2\pi i d/d\tau$, one obtains a non-autonomous Hamiltonian system:

$$
\begin{aligned}
2\pi i \frac{dq_j}{d\tau} &= \{q_j, \mathcal{H}\} = p_j, \\
2\pi i \frac{dp_j}{d\tau} &= \{p_j, \mathcal{H}\} = -g^2 \sum_{k\neq j} \wp'(q_j - q_k).
\end{aligned} \tag{2.25}
$$

We now demonstrate that this gives an isomonodromic system on the torus E_τ. A key is the following Lax equation:

Proposition 3 *$L(z)$ and $M(z)$ satisfy the Lax equation*

$$
2\pi i \frac{\partial L(z)}{\partial \tau} + \frac{\partial M(z)}{\partial z} = [L(z), M(z)]. \tag{2.26}
$$

Proof. Let us notice that the right hand side of the isospectral Lax equation is in fact the Poisson bracket of $L(z)$ and the Hamiltonian:

$$
[L(z), M(z)] = \frac{\partial L(z)}{\partial t} = \{L(z), \mathcal{H}\}. \tag{2.27}
$$

Since the phase space and the Hamiltonian are the same as those of the original system, the relation $[L(z), M(z)] = \{L(z), \mathcal{H}\}$ persists in the present setup. Thus the right hand side of the Lax equation can be written

$$
\begin{aligned}
[L(z), M(z)] &= \{L(z), \mathcal{H}\} \\
&= \sum_{j=1}^{\ell} \{p_j, \mathcal{H}\} E_{jj} + ig \sum_{j\neq k} \{q_j - q_k, \mathcal{H}\} y(q_j - q_k, z) E_{jk}.
\end{aligned} \tag{2.28}
$$

On the other hand,

$$
\begin{aligned}
2\pi i \frac{\partial L(z)}{\partial \tau} + \frac{\partial M(z)}{\partial z} &= \sum_{j=1}^{\ell} 2\pi i \frac{dp_j}{d\tau} E_{jj} \\
&\quad + ig \sum_{j\neq k} 2\pi i \left(\frac{dq_j}{d\tau} - \frac{dq_k}{d\tau} \right) y(q_j - q_k, z) E_{jk} \\
&\quad + ig \sum_{j\neq k} \left(2\pi i \frac{\partial x(u,z)}{\partial \tau} + \frac{\partial y(u,z)}{\partial z} \right)_{u = q_j - q_k} E_{jk}.
\end{aligned} \tag{2.29}
$$

The last sum vanishes because of the "heat equation" (2.18). The other part coincides, term-by-term, with the above expression of the commutator $[L(z), M(z)]$. *Q.E.D.*

This Lax equation enables us to interpret the non-autonomous Hamiltonian system as an isomonodromic system on the torus E_τ. The Lax equation is nothing but the Frobenius integrability condition of a linear system of the form

$$\frac{\partial Y(z)}{\partial z} = L(z)Y(z), \quad 2\pi i \frac{\partial Y(z)}{\partial \tau} + M(z)Y(z) = 0. \tag{2.30}$$

The first equation is an ordinary differential equation on the torus E_τ, and has a regular singular point at $z = 0$. Analytic continuation of the solution around this singular point yields a monodromy matrix Γ_0. Besides this local monodromy matrix, there are global monodromy matrices Γ_α and Γ_β that arise in analytic continuation along the α ($z \to z+1$) and β ($z \to z+\tau$) cycles. The second equation of the above linear system implies that these monodromy matrices are left invariant as τ varies.

Let us specify this observation in more detail. The situation is more complicated than isomonodromic systems on the Riemann sphere: The monodromy of $L(z)$ and $M(z)$ themselves are non-trivial,

$$\begin{aligned} &L(z+1) = L(z), \quad M(z+1) = M(z), \\ &L(z+\tau) = e^{2\pi i Q} L(z) e^{-2\pi i Q}, \\ &M(z+\tau) = e^{2\pi i Q}(M(z) + 2\pi i L(z))e^{-2\pi i Q} - 2\pi i P, \end{aligned} \tag{2.31}$$

where $Q = \sum_{j=1}^{\ell} q_j E_{jj}$ and $P = \sum_{j=1}^{\ell} p_j E_{jj}$. These relations are a consequence of the quasi-periodicity of $x(u,z)$, $y(u,z)$ and $\rho(z)$. The monodromy of $L(z)$ implies that $Y(z)$ has to be treated as a section of a non-trivial $GL(\ell,\mathbb{C})$-bundle (or $SL(\ell,\mathbb{C})$-bundle, if we take the center of mass frame with $\sum_{j=1}^{\ell} p_j = 0$) on the torus E_τ. The monodromy matrices Γ_0, Γ_α and Γ_β thus arise as follows:

$$Y(ze^{2\pi i}) = Y(z)\Gamma_0, \quad Y(z+1) = Y(z)\Gamma_\alpha, \quad Y(z+\tau) = e^{2\pi i Q} Y(z)\Gamma_\beta. \tag{2.32}$$

Note that the exponential factor in the last relation reflects the non-trivial monodromy of $L(z)$ along the β-cycle. Having this monodromy structure of $Y(z)$, one can deduce the following fundamental observation:

Proposition 4 *The monodromy matrices do not depend on τ, i.e.,*

$$\frac{d\Gamma_0}{d\tau} = \frac{d\Gamma_\alpha}{d\tau} = \frac{d\Gamma_\beta}{d\tau} = 0. \tag{2.33}$$

Proof. Let us rewrite the second equation of the linear system as

$$M(z) = -2\pi i \frac{\partial Y(z)}{\partial \tau} Y(z)^{-1}, \tag{2.34}$$

and examine the implication of the monodromy structure of $Y(z)$ noted above. This leads to the following relations:

$$\begin{aligned} M(ze^{2\pi i}) &= M(z) - 2\pi i Y(z) \frac{\partial \Gamma_0}{\partial \tau} \Gamma_0^{-1} Y(z)^{-1}, \\ M(z+1) &= M(z) - 2\pi i Y(z) \frac{\partial \Gamma_\alpha}{\partial \tau} \Gamma_\alpha^{-1} Y(z)^{-1}, \\ M(z+\tau) &= e^{2\pi i Q}(M(z) + 2\pi i L(z)) e^{-2\pi i Q} - 2\pi i P \\ &\quad -2\pi i Y(z) \frac{\partial \Gamma_\beta}{\partial \tau} \Gamma_\beta^{-1} Y(z)^{-1}. \end{aligned} \tag{2.35}$$

(We have used the relation $2\pi i dQ/d\tau = P$.) These relations are consistent with the aforementioned monodromy structure of $M(z)$ if and only if the monodromy matrices of $Y(z)$ are independent of τ. *Q.E.D.*

3 Elliptic Calogero-Moser Systems Based on Root Systems

Here we consider the elliptic Calogero-Moser systems associated with a general irreducible (but not necessary reduced) root system Δ.

In the following, the root system Δ is assumed to be realized in an ℓ-dimensional Euclidean space $M = \mathbb{R}^\ell$. Let $x \cdot y$ denote the inner product of two vectors in M and its bilinear extension to the complexification $M^{\mathbb{C}} = M \otimes_{\mathbb{R}} \mathbb{C}$. The dual space $M^* = Hom(M, \mathbb{R})$ of M is identified with M by this inner product. Each element $\alpha \in \Delta$ induces a reflection (the Weyl reflection) $s_\alpha(x) = x - (2\alpha \cdot x/\alpha \cdot \alpha)\alpha$. This gives a representation of the Weyl group $W(\Delta)$ on M. The root system Δ is invariant under the action of this Weyl group.

The elliptic Calogero-Moser system associated with the root system Δ is a Hamiltonian system on $M \times M$ (or its complexification $M^{\mathbb{C}} \times M^{\mathbb{C}}$). The orthognal coordinates $(q,p) = (q_1, \cdots, q_\ell, p_1, \cdots, p_\ell)$ of $M \times M$ give canonical coordinates and momenta with the Poisson

brackets

$$\{q_j, p_k\} = \delta_{jk}, \quad \{p_j, p_k\} = \{q_j, q_k\} = 0. \tag{3.1}$$

3.1 Simply laced models

We first consider the case of simply laced ($A_{\ell-1}$, D_ℓ and E_ℓ) root systems. The associated elliptic Calogero-Moser system is defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} p \cdot p + \frac{g^2}{2} \sum_{\alpha \in \Delta} \wp(\alpha \cdot q). \tag{3.2}$$

Here g is a coupling constant, and $\wp(u)$ the Weierstrass $\wp$ function with primitive periods 1 and τ. The equations of motion can be written

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = -\frac{g^2}{2} \sum_{\alpha \in \Delta} \wp'(\alpha \cdot q)\alpha. \tag{3.3}$$

We first review the "root type" Lax pair of Bordner et al. for these models [12], then explain how to convert these isospectral systems to isomonodromic systems.

3.1.1 Root type Lax pair

The "root type" Lax pair for these simply laced models are $\Delta \times \Delta$ matrices, i.e., matrices whose rows and columns are indexed by the root system Δ. They are made of three parts:

$$L(z) = P + X_1(z) + X_2(z), \quad M(z) = D + Y_1(z) + Y_2(z). \tag{3.4}$$

P and D are diagonal matrices,

$$P_{\beta\gamma} = p \cdot \beta \delta_{\beta\gamma}, \quad D_{\beta\gamma} = D_\beta \delta_{\beta\gamma} \quad (\beta, \gamma \in \Delta), \tag{3.5}$$

and the diagonal elements D_β of D are given by

$$D_\beta = ig\wp(\beta \cdot q) + ig \sum_{\gamma \in \Delta, \beta \cdot \gamma = 1} \wp(\gamma \cdot q). \tag{3.6}$$

$X_1(z)$, etc. are diagonal-free matrices of the form

$$X_1(z) \;\; = \;\; ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, z) E(\alpha),$$

$$
\begin{aligned}
X_2(z) &= 2ig \sum_{\alpha\in\Delta} x(\alpha\cdot q, 2z) E(2\alpha), \\
Y_1(z) &= ig \sum_{\alpha\in\Delta} y(\alpha\cdot q, z) E(\alpha), \\
Y_2(z) &= ig \sum_{\alpha\in\Delta} y(\alpha\cdot q, 2z) E(2\alpha),
\end{aligned} \tag{3.7}
$$

where $x(u,z)$ and $y(u,z)$ are the same as the functions used in the previous section, and $E(\alpha)$ and $E(2\alpha)$ are $\Delta\times\Delta$ matrices of the form

$$
E(\alpha)_{\beta\gamma} = \delta_{\alpha,\beta-\gamma}, \quad E(2\alpha)_{\beta\gamma} = \delta_{2\alpha,\beta-\gamma} \quad (\beta,\gamma\in\Delta). \tag{3.8}
$$

(We have slightly modified the notation of Bordner et al: $x(u,2z)$, $y(u,2z)$ and $E(2\alpha)$ amount to $x_d(u,z)$, $y_d(u,z)$ and $E_d(\alpha)$ in their notation.)

These matrices satisfy the Lax equation

$$
\frac{\partial L(z)}{\partial t} = [L(z), M(z)] \tag{3.9}
$$

under the equations of motions. The traces $\mathrm{Tr}\, L(z)^k$, $k = 2,3,\cdots$, of powers of $L(z)$ are conserved, and an involutive set of conserved quantities can be extracted from these traces. The Hamiltonian itself can be reproduced from the quadratic trace $\mathrm{Tr}\, L(z)^2$. We refer the details of these results to the paper of Bordner et al. [12]. The choice of $x(u,z)$ and $y(u,z)$ is irrelevant in this case, too.

Thus, in particular, the $A_{\ell-1}$ model turns out to have at least two distinct Lax pairs — the Lax pair of $\ell\times\ell$ matrices realized in the vector representation of $sl(\ell)$, and the Lax pair of $\ell(\ell-1)\times\ell(\ell-1)$ matrices based on the $A_{\ell-1}$ root system. This is also the case for the other simply laced root systems. Bordner et al. call the Lax pairs of the first type the "minimal type", because they are realized in a minimal representation of the associated (not necessary simply laced) Lie algebra. It should be noted that the "root type" Lax pairs do not possess a Lie algebraic structure; unlike the usual root basis of simple Lie algebras, the matrices $E(\alpha)$ and $E(2\alpha)$ are not closed under the Lie bracket.

3.1.2 Isomonodromic system

The prescription for constructing an isomonodromic analogue is the same as the previous case, namely, to replace $d/dt \to 2\pi i d/d\tau$. This converts the equations of motion of the

elliptic Calogero-Moser system to the non-autonomous system

$$2\pi i\frac{dq}{dt} = p, \quad 2\pi i\frac{dp}{dt} = -\frac{g^2}{2}\sum_{\alpha\in\Delta}\wp'(\alpha\cdot q)\alpha. \tag{3.10}$$

Let $x(u,z)$ be the function defined in (2.14), and $y(u,z)$ its u-derivative. The following are the keys to an isomonodromic interpretation.

Proposition 5 *1. $L(z)$ and $K(z)$ satisfy the Lax equation*

$$2\pi i\frac{\partial L(z)}{\partial\tau} + \frac{\partial M(z)}{\partial z} = [L(z), M(z)]. \tag{3.11}$$

2. $L(z)$ and $M(z)$ have the following monodromy property:

$$\begin{aligned}
&L(z+1) = L(z), \quad M(z+1) = M(z),\\
&L(z+\tau) = e^{2\pi iQ}L(z)e^{-2\pi iQ},\\
&M(z+\tau) = e^{2\pi iQ}(M(z) + 2\pi iL(z))e^{-2\pi iQ} - 2\pi iP,
\end{aligned}\tag{3.12}$$

where Q is the diagonal matrix with matrix elements $Q_{\beta\gamma} = q\cdot\beta\delta_{\beta\gamma}$.

Proof. The proof is almost the same as the proof for the isomonodromic Lax pair of the $A_{\ell-1}$ model in the vector representation. Let us first verify the Lax equation. The right hand side of the Lax equation can be written

$$\begin{aligned}
[L(z), M(z)] &= \{P,\mathcal{H}\} + ig\sum_{\alpha\in\Delta}\{\alpha\cdot q,\mathcal{H}\}y(\alpha\cdot q, z)E(\alpha)\\
&\quad +2ig\sum_{\alpha\in\Delta}\{\alpha\cdot q,\mathcal{H}\}y(\alpha\cdot q, 2z)E(2\alpha).
\end{aligned}\tag{3.13}$$

On the other hand,

$$\begin{aligned}
2\pi i\frac{\partial L(z)}{\partial\tau} + \frac{\partial M(z)}{\partial z} &= 2\pi i\frac{\partial P}{\partial\tau} + ig\sum_{\alpha\in\Delta}2\pi i\frac{\partial\alpha\cdot q}{\partial\tau}y(\alpha\cdot q, z)E(\alpha)\\
&\quad +2ig\sum_{\alpha\in\Delta}2\pi i\frac{\partial\alpha\cdot q}{\partial\tau}y(\alpha\cdot q, 2z)E(2\alpha)\\
&\quad +ig\sum_{\alpha\in\Delta}\left(2\pi i\frac{\partial x(u,z)}{\partial\tau} + \frac{\partial y(u,z)}{\partial z}\right)_{u=\alpha\cdot q}E(\alpha)\\
&\quad +2ig\sum_{\alpha\in\Delta}\left(4\pi i\frac{\partial x(u,2z)}{\partial\tau} + \frac{\partial y(u,2z)}{\partial z}\right)_{u=\alpha\cdot q}E(2\alpha).
\end{aligned}\tag{3.14}$$

The last two sums vanish because of (2.18). The other part coincides by the equations of motion. Thus we obtain the Lax equation. Let us next consider the monodromy of $L(z)$ and $M(z)$. Note the commutation relations

$$[Q, E(\alpha)] = q \cdot \alpha E(\alpha), \quad [Q, E(2\alpha)] = 2q \cdot \alpha E(2\alpha), \tag{3.15}$$

which can be exponentiated as follows:

$$e^{2\pi i Q} E(\alpha) e^{-2\pi i Q} = e^{2\pi i q \cdot \alpha} E(\alpha), \quad e^{2\pi i Q} E(2\alpha) e^{-2\pi i Q} = e^{4\pi i q \cdot \alpha} E(2\alpha). \tag{3.16}$$

The monodromy property of $L(z)$ and $M(z)$ can be derived from these relations and the quasi-periodicity of $x(u, z)$ and $y(u, z)$. *Q.E.D.*

The rest is parallel to the case in the previous section. The only difference is that the ordinary differential equation

$$\frac{dY(z)}{dz} = L(z)Y(z) \tag{3.17}$$

on the torus E_τ has *four* regular singular points at $z = 0, \omega_1, \omega_2, \omega_3$. The latter three singular points originates in $X_2(z)$. Let Γ_a ($a = 0, 1, 2, 3$) denote the monodromy matrices in analytic continuation of $Y(z)$ around these four points. The Lax equation implies that these local monodromy matrices and the two global ones Γ_α and Γ_β are independent of τ:

$$\frac{\partial \Gamma_0}{\partial \tau} = \cdots = \frac{\partial \Gamma_3}{\partial \tau} = \frac{\partial \Gamma_\alpha}{\partial \tau} = \frac{\partial \Gamma_\beta}{\partial \tau} = 0. \tag{3.18}$$

3.2 Non-simply laced models

The elliptic Calogero-Moser system associated with a non-simply laced (B_ℓ, C_ℓ, F_4, G_2 and BC_ℓ) root systems can have several independent coupling constants, one for each Weyl group orbit in the root system. The root type Lax pairs are extended to the non-simply laced cases by Bordner et al. [13]. As they pointed out, one can construct a different root type Lax pair for each Weyl group orbit of the root system. Thus the B_ℓ, C_ℓ, F_4 and G_2 models have, respectively, two distinct Lax pairs based on the orbits of long and short roots, whereas the BC_ℓ model has three based on the orbits of long, middle, and short roots. Note that each Weyl group orbit consists of roots of the same length.

Although all the non-simply laced models can be treated in the same way, let us illustrate our construction of isomonodromic systems for the BC_ℓ model. This is also intended to be a prototype of the case that we shall consider in the next subsection.

3.2.1 BC_ℓ model

The BC_ℓ root system can be realized in $M = \mathbb{R}^\ell$:

$$\begin{aligned}
\Delta(BC_\ell) &= \Delta_l \cup \Delta_m \cup \Delta_s, \\
\Delta_l &= \{\pm 2e_j \mid 1 \le j \le \ell\} \quad \text{(long roots)}, \\
\Delta_m &= \{\pm e_j \pm e_k \mid j \ne k\} \quad \text{(middle roots)}, \\
\Delta_s &= \{\pm e_j \mid 1 \le j \le \ell\} \quad \text{(short roots)},
\end{aligned} \tag{3.19}$$

where $e_1, \cdots, e_\ell$ are the standard orthonormal basis of $\mathbb{R}^\ell$. Δ_l, Δ_m and Δ_s give the three Weyl group orbits.

The Hamiltonian of the BC_ℓ model takes the form

$$\mathcal{H} = \frac{1}{2} p \cdot p + \frac{g_m^2}{2} \sum_{\alpha \in \Delta_m} \wp(\alpha \cdot q) + \frac{g_l^2}{4} \sum_{\alpha \in \Delta_l} \wp(\alpha \cdot q) + \tilde{g}_s^2 \sum_{\alpha \in \Delta_s} \wp(\alpha \cdot q). \tag{3.20}$$

The equations of motion can be written

$$\begin{aligned}
\frac{dq}{d\tau} &= p, \\
\frac{dp}{d\tau} &= -\frac{g_m^2}{2} \sum_{\alpha \in \Delta_m} \wp'(\alpha \cdot q)\alpha - \frac{g_l^2}{4} \sum_{\alpha \in \Delta_l} \wp'(\alpha \cdot q)\alpha - \tilde{g}_s^2 \sum_{\alpha \in \Delta_s} \wp'(\alpha \cdot q)\alpha.
\end{aligned} \tag{3.21}$$

g_m, g_l and $\tilde{g}_s$ are three independent coupling constants. $\tilde{g}_s$ is a modified ("renormalized" in the terminology of Bordner et al.) coupling constant connected with a more fundamental ("bare", so to speak) coupling constant g_s as

$$\tilde{g}_s^2 = g_s^2 + \frac{g_s g_l}{2}. \tag{3.22}$$

The "bare" coupling constant appears in the construction of a Lax pair.

3.2.2 Root type Lax pair for BC_ℓ model

As mentioned above, there are at least three root type Lax pairs based on the three Weyl group orbits Δ_m, Δ_l and Δ_s. Bordner et al. constructed only one of them, namely, a Lax pair based on Δ_m. Here we present a Lax pair based on Δ_s. This is a $2\ell \times 2\ell$ system, much smaller than the Lax pair based on Δ_m, and presumably more suitable for studying the associated isomonodromic deformations.

The Lax pair are indexed by Δ_s and take the following form:

$$\begin{aligned} L(z) &= P + X_1(z) + X_2(z) + X_3(z), \\ M(z) &= D + Y_1(z) + Y_2(z) + Y_3(z). \end{aligned} \tag{3.23}$$

P and D are diagonal matrices,

$$P_{\beta\gamma} = p \cdot \beta \delta_{\beta\gamma}, \quad D_{\beta\gamma} = D_\beta \delta_{\beta\gamma} \quad (\beta, \gamma \in \Delta_s), \tag{3.24}$$

and the diagonal elements of D are given by

$$D_\beta = i g_m \sum_{\gamma \in \Delta_s, \beta\cdot\gamma = 1} \wp(\gamma \cdot q) + i g_l \wp(2\beta \cdot q) + i g_s \wp(\beta \cdot q). \tag{3.25}$$

$X_1(z)$, etc. are diagonal-free matrices of the form

$$\begin{aligned} X_1(z) &= i g_m \sum_{\alpha \in \Delta_m} x(\alpha \cdot q, z) E(\alpha), \\ X_2(z) &= i g_l \sum_{\alpha \in \Delta_l} x(\alpha \cdot q, z) E(\alpha), \\ X_3(z) &= 2 i g_s \sum_{\alpha \in \Delta_s} x(\alpha \cdot q, 2z) E(2\alpha), \\ Y_1(z) &= i g_m \sum_{\alpha \in \Delta_m} y(\alpha \cdot q, z) E(\alpha), \\ Y_2(z) &= i g_l \sum_{\alpha \in \Delta_l} y(\alpha \cdot q, z) E(\alpha), \\ Y_3(z) &= i g_s \sum_{\alpha \in \Delta_s} y(\alpha \cdot q, 2z) E(2\alpha), \end{aligned} \tag{3.26}$$

where

$$E(\alpha)_{\beta\gamma} = \delta_{\alpha, \beta-\gamma}, \quad E(2\alpha)_{\beta\gamma} = \delta_{2\alpha, \beta-\gamma} \quad (\beta, \gamma \in \Delta_s). \tag{3.27}$$

This Lax pair is a specialization of the Lax pair for the extended twisted model that we shall present in the next subsection.

3.2.3 Isomonodromic system

This system, too, can be converted to an isomonodromic system by replacing $d/dt \to 2\pi i d/d\tau$. The equations of motion are a non-autonomous system of the form

$$\begin{aligned} 2\pi i \frac{dq}{d\tau} &= p, \\ 2\pi i \frac{dp}{d\tau} &= -\frac{g_m^2}{2} \sum_{\alpha \in \Delta_m} \wp'(\alpha \cdot q)\alpha - \frac{g_l^2}{4} \sum_{\alpha \in \Delta_l} \wp'(\alpha \cdot q)\alpha - \tilde{g}_s^2 \sum_{\alpha \in \Delta_s} \wp'(\alpha \cdot q)\alpha. \end{aligned} \tag{3.28}$$

The following can be verified just as in the case of simply lased models:

1. $L(z)$ and $M(z)$ satisfy the Lax equation

$$2\pi i\frac{\partial L(z)}{\partial\tau}+\frac{\partial M(z)}{\partial z}=[L(z),M(z)]. \tag{3.29}$$

2. $L(z)$ and $M(z)$ have the following monodromy property:

$$\begin{aligned}
&L(z+1)=L(z), \quad M(z+1)=M(z),\\
&L(z+\tau)=e^{2\pi iQ}L(z)e^{-2\pi iQ},\\
&M(z+\tau)=e^{2\pi iQ}(M(z)+2\pi iL(z))e^{-2\pi iQ}-2\pi iP.
\end{aligned} \tag{3.30}$$

The interpretation of this Lax equation, too, is parallel to the simply laced models. The ordinary differential equation

$$\frac{dY(z)}{dz}=L(z)Y(z) \tag{3.31}$$

on the torus E_τ has four regular singular points at $z=0,\omega_1,\omega_2,\omega_3$. The local monodromy matrices Γ_a $(a=0,1,2,3)$ at these points and the global monodromy matrices Γ_α and Γ_β are invariant as τ varies.

3.3 Twisted and extended twisted models

We now proceed to the "twisted" and "extended twisted" models. The Hamiltonian of the untwisted models can be generally written

$$\mathcal{H}=\frac{1}{2}p\cdot p+\frac{1}{2}\sum_{\alpha\in\Delta}g_{|\alpha|}^2\wp(\alpha\cdot q). \tag{3.32}$$

The twisted models, introduced by D'Hoker and Phong [15] for non-simply laced root systems, are defined by a Hamiltonian of the form

$$\mathcal{H}=\frac{1}{2}p\cdot p+\frac{1}{2}\sum_{\alpha\in\Delta}g_{|\alpha|}^2\wp_{\nu(\alpha)}(\alpha\cdot q), \tag{3.33}$$

where $\wp_{\nu(\alpha)}(u)$ are the $\wp$-functions with suitably rescaled primitive periods. D'Hoker and Phong proved the integrability of those twisted models by constructing a Lax pair in a representation of the associated Lie algebra. Bordner and Sasaki [14] proposed an

alternative approach based on root systems rather than Lie algebras, and pointed out that the twisted model of the B_ℓ, C_ℓ and BC_ℓ types can be further extended. The extended twisted models have one (for the B_ℓ and C_ℓ models) or two (for the BC_ℓ model) extra types of elliptic potentials.

Our construction of isomonodromic systems can be extended to the twisted and extended twisted models. We illustrate this result, just as in the previous subsection, for the BC_ℓ model. As Bordner and Sasaki noted, the extended twisted BC_ℓ model is made of five different types of elliptic potentials, and coincides with the Inozemtsev system [6].

3.3.1 Extended twisted BC_ℓ model

The extended twisted BC_ℓ model is defined by the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}p\cdot p + \frac{g_m^2}{2}\sum_{\alpha\in\Delta_m}\wp(\alpha\cdot q) + \frac{g_{l1}^2}{4}\sum_{\alpha\in\Delta_l}\wp(\alpha\cdot q) + \frac{\tilde{g}_{l2}^2}{4}\sum_{\alpha\in\Delta_l}\wp^{(2)}(\alpha\cdot q) \\ &\quad + \tilde{g}_{s1}^2\sum_{\alpha\in\Delta_s}\wp(\alpha\cdot q) + \tilde{g}_{s2}^2\sum_{\alpha\in\Delta_s}\wp^{(1/2)}(\alpha\cdot q). \end{aligned} \tag{3.34}$$

$\tilde{g}_{l2}$, $\tilde{g}_{s1}$ and $\tilde{g}_{s2}$ are "renormalized" coupling constants, which are related to unrenormalized coupling constants g_{l2}, g_{s1} and g_{s2} as follows:

$$\begin{aligned} \tilde{g}_{l2}^2 &= g_{l2}^2 + 2g_{l1}g_{l2}, \\ \tilde{g}_{s1}^2 &= g_{s1}^2 + 2g_{s1}g_{s2} + \frac{1}{2}(g_{s1}g_{l1} + g_{s1}g_{l2} + g_{s2}g_{l2}), \\ \tilde{g}_{s2}^2 &= g_{s2}^2 + \frac{g_{s2}g_{l1}}{2}. \end{aligned} \tag{3.35}$$

$\wp^{(1/2)}$ and $\wp^{(2)}$ are the $\wp$ functions with rescaled primitive periods:

$$\wp^{(1/2)}(u) = \wp(u \mid \frac{1}{2}, \tau), \quad \wp^{(2)}(u) = \wp(u \mid 2, \tau). \tag{3.36}$$

(This Hamiltonian is slightly different from the Hamiltonian of Bordner and Sasaki, though the contents are essentially the same. With this modification, this model reduces to the untwisted BC_ℓ model as $g_{l2} \to 0$ and $g_{s2} \to 0$.)

3.3.2 Root type Lax pair for extended twisted BC_ℓ model

One can construct, like the untwisted model, three different root type Lax pairs can be constructed based on the three Weyl group orbits Δ_m, Δ_l and Δ_s. The Lax pair based

on Δ_m is presented by Bordner and Sasaki. The Lax pair based on Δ_s can be obtained by modifying the Lax pair for the untwisted BC_ℓ model as follows.

The Lax pair $L(z)$ and $M(z)$ are indexed by Δ_s and made of four parts,

$$
\begin{aligned}
L(z) &= P + X_1(z) + X_2(z) + X_3(z), \\
M(z) &= D + Y_1(z) + Y_2(z) + Y_3(z).
\end{aligned} \tag{3.37}
$$

The diagonal matrix P is the same as the P in the untwisted model. The diagonal matrices of D are given by

$$
\begin{aligned}
D_\beta &= ig_m \sum_{\gamma\in\Delta_m, \beta\cdot\gamma=1} \wp(\gamma\cdot q) + ig_{l1}\wp(2\beta\cdot q) + ig_{l2}\wp^{(2)}(2\beta\cdot q) \\
&\quad + ig_{s1}\wp(\beta\cdot q) + ig_{s2}\wp^{(1/2)}(\beta\cdot q).
\end{aligned} \tag{3.38}
$$

$X_1(z)$ and $Y_1(z)$ are the same as those for the untwisted model. The other matrices take the following form:

$$
\begin{aligned}
X_2(z) &= \sum_{\alpha\in\Delta_l} \Big(ig_{l1}x(\alpha\cdot q, z) + ig_{l2}x^{(2)}(\alpha\cdot q, z) \Big) E(\alpha), \\
X_3(z) &= \sum_{\alpha\in\Delta_s} \Big(2ig_{s1}x(\alpha\cdot q, 2z) + 2ig_{s2}x^{(1/2)}(\alpha\cdot q, 2z) \Big) E(2\alpha), \\
Y_2(z) &= \sum_{\alpha\in\Delta_l} \Big(ig_{l1}y(\alpha\cdot q, z) + ig_{l2}y^{(2)}(\alpha\cdot q, z) \Big) E(\alpha), \\
Y_3(z) &= \sum_{\alpha\in\Delta_s} \Big(ig_{s1}y(\alpha\cdot q, 2z) + ig_{s2}y^{(1/2)}(\alpha\cdot q, 2z) \Big) E(2\alpha).
\end{aligned} \tag{3.39}
$$

This Lax pair reduces to the Lax pair of the untwisted model if $g_{l2}=0$ and $g_{s2}=0$.

The new objects arising here are the functions $x^{(1/2)}(u,z)$, $x^{(2)}(u,z)$ and their u-derivatives

$$
y^{(1/2)}(u,z) = \frac{\partial x^{(1/2)}(u,z)}{\partial u}, \quad y^{(2)}(u,z) = \frac{\partial x^{(2)}(u,z)}{\partial u}. \tag{3.40}
$$

For the consistency of the Lax equation

$$
\frac{\partial L(z)}{\partial t} = [L(z), M(z)], \tag{3.41}
$$

these functions have to satisfy several functional equations. D'Hoker and Phong [15] and Bordner and Sasaki [14] use a set of functions based on the Weierstrass sigma functions.

We use the function $x(u,z) = x(u,z \mid \tau)$ defined in (2.14) and its modifications

$$\begin{aligned}
x^{(1/2)}(u,z) &= 2x(2u,z \mid 2\tau) = \frac{2\theta_1(z-2u \mid 2\tau)\theta_1'(0 \mid 2\tau)}{\theta_1(z \mid 2\tau)\theta_1(2u \mid 2\tau)}, \\
x^{(2)}(u,z) &= \frac{1}{2}x(\frac{u}{2}, z \mid \frac{\tau}{2}) = \frac{\theta_1(z-\frac{u}{2} \mid \frac{\tau}{2})\theta_1'(0 \mid \frac{\tau}{2})}{2\theta_1(z \mid \frac{\tau}{2})\theta_1(\frac{u}{2} \mid \frac{\tau}{2})}.
\end{aligned} \tag{3.42}$$

These functions $x^{(1/2)}(u,z)$ and $x^{(2)}(u,z)$, too, satisfy $1+2$-dimensional "heat equations" of the form

$$\begin{aligned}
2\pi i \frac{\partial x^{(1/2)}(u,z)}{\partial \tau} + \frac{\partial^2 x^{(1/2)}(u,z)}{\partial u \partial z} &= 0, \\
2\pi i \frac{\partial x^{(2)}(u,z)}{\partial \tau} + \frac{\partial^2 x^{(2)}(u,z)}{\partial u \partial z} &= 0.
\end{aligned} \tag{3.43}$$

The functional identities for these functions and the proof of the Lax equation are presented in Appendices B and C.

3.3.3 Isomonodromic system

Replacing $d/dt \to 2\pi i d/d\tau$, we obtain a non-autonomous Hamiltonian system with the same Hamiltonian. The isomonodromic interpretation of this non-autonomous system is again based on the following two observations:

1. $L(z)$ and $M(z)$ satisfy the Lax equation
$$2\pi i \frac{\partial L(z)}{\partial \tau} + \frac{\partial M(z)}{\partial z} = [L(z), M(z)]. \tag{3.44}$$

2. The monodromy of $L(z)$ and $M(z)$ is the same as the monodromy of the Lax pair for the untwisted model:
$$\begin{aligned}
&L(z+1) = L(z), \quad M(z+1) = M(z), \\
&L(z+\tau) = e^{2\pi i Q} L(z) e^{-2\pi i Q}, \\
&M(z+\tau) = e^{2\pi i Q}(M(z) + 2\pi i L(z)) e^{-2\pi i Q} - 2\pi i P.
\end{aligned} \tag{3.45}$$

The ordinary differential equation defined on the torus E_τ by the matrix $L(z)$ has four regular singular points at $u = 0, \omega_1, \omega_2, \omega_3$. The Lax equation and the monodromy of $L(z)$ and $M(z)$ ensure that the local monodromy matrices Γ_a ($a = 0,1,2,3$) and the global monodromy matrices Γ_α and Γ_β are independent of τ.

3.3.4 Relation to Inozemtsev system

The final task is to clarify the relation to the Inozemtsev system. In terms of the orthogonal coordinates $q_j = q \cdot e_j$ and $p_j = p \cdot e_j$ $(j = 1, \cdots, \ell)$, the aforementioned Hamiltonian can be written

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\sum_{j=1}^{\ell} p_j^2 + \frac{g_m^2}{2}\sum_{\epsilon,\epsilon'=\pm 1}\sum_{j\neq k}\wp(\epsilon q_j + \epsilon' q_k) + \frac{g_{l1}^2}{2}\sum_{j=1}^{\ell}\wp(2q_j) \\ &\quad + \frac{\tilde{g}_{l2}^2}{2}\sum_{j=1}^{\ell}\wp^{(2)}(2q_j) + 2\tilde{g}_{s1}^2\sum_{j=1}^{\ell}\wp(q_j) + 2\tilde{g}_{s2}^2\sum_{j=1}^{\ell}\wp^{(1/2)}(q_j). \end{aligned} \tag{3.46}$$

One can rewrite this Hamiltonian using the identities

$$\begin{aligned}\wp(2u) &= \frac{1}{4}\wp(u) + \frac{1}{4}\wp(u+\omega_1) + \frac{1}{4}\wp(u+\omega_2) + \frac{1}{4}\wp(u+\omega_3), \\ \wp^{(1/2)}(u) &= \wp(u) + \wp(u+\omega_1) - \wp(\omega_1), \\ \wp^{(2)}(2u) &= \frac{1}{4}\wp(u) + \frac{1}{4}\wp(u+\omega_3) - \frac{1}{4}\wp(\omega_3). \end{aligned} \tag{3.47}$$

The outcome is, up to a term $h(\tau)$ depending on τ only, the Inozemtsev Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{\ell} p_j^2 + \frac{g_m^2}{2}\sum_{\epsilon,\epsilon'=\pm 1}\sum_{j\neq k}\wp(\epsilon q_j + \epsilon' q_k) + \sum_{j=1}^{\ell}\sum_{a=0}^{3} g_a^2\wp(q_j+\omega_a) + h(\tau). \tag{3.48}$$

The coupling constants g_a $(a = 0, 1, 2, 3)$ are given by

$$\begin{aligned} g_0^2 &= \frac{1}{8}(g_{l1}^2 + \tilde{g}_{l2}^2) + 2(\tilde{g}_{s1}^2 + \tilde{g}_{s2}^2), \qquad g_1^2 = \frac{g_{l1}^2}{8} + 2\tilde{g}_{s2}^2, \\ g_2^2 &= \frac{g_{l1}^2}{8}, \qquad g_3^2 = \frac{1}{8}(g_{l1}^2 + \tilde{g}_{l2}^2). \end{aligned} \tag{3.49}$$

4 Spin Generalization of Elliptic Calogero-Moser Systems

"Spin generalization" is a generalization of the elliptic Calogero-Moser systems coupled to spin degrees of freedom. Such a spin generalization is characterized by a simple Lie algebra rather than a root system. The (classical) spin variables take values in the dual space $\mathfrak{g}^*$, or a coadjoint orbit therein, of the Lie algebra $\mathfrak{g}$. We shall first examine the $sl(\ell)$ model as a prototype, then proceed to the models based on a general simple Lie algebra.

4.1 Spin generalization for $sl(\ell)$

The $sl(\ell)$ spin generalization was first introduced by Krichever et al. [19]. They obtained the spin generalization, just like the spinless case [5], via the pole dynamics of the matrix KP hierarchy.

4.1.1 Hamiltonian formalism

This model is a constrained Hamiltonian system. The Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{\ell} p_j^2 - \frac{1}{2}\sum_{j\neq k} \wp(q_j - q_k) F_{jk} F_{kj}. \tag{4.1}$$

Here q_j and p_j $(j = 1, \cdots, \ell)$ are the canonical coordinates and momenta of the Calogero-Moser particles, and F_{jk} $(j, k = 1, \cdots, \ell)$ a set of classical $sl(\ell)$ spin variables, whose Poisson brackets are determined by the Kostant-Kirillov Poisson structure on the dual space of $sl(\ell)$:

$$\{F_{jk}, F_{mn}\} = \delta_{mk} F_{jn} - \delta_{jn} F_{mk}. \tag{4.2}$$

The equations of motion can be written

$$\begin{aligned}
\frac{dq_j}{dt} &= p_j, \quad \frac{dp_j}{dt} = \sum_{k\neq j} \wp'(q_j - q_k) F_{jk} F_{kj}, \\
\frac{dF_{jk}}{dt} &= -\sum_{m\neq j} \wp(q_j - q_m) F_{jm} + \sum_{m\neq k} \wp(q_m - q_k) F_{mk} \\
&\quad -\wp(q_j - q_k)(F_{jj} - F_{kk}).
\end{aligned} \tag{4.3}$$

In particular, the diagonal elements F_{jj} of the spin variables are conserved quantities: $dF_{jj}/dt = 0$. Although the Hamiltonian does not contain the diagonal elements explicitly, they do appear in the equations of motion. We now put the constraints

$$F_{jj} = 0 \quad (j = 1, \cdots, \ell). \tag{4.4}$$

These constraints ensure the integrability. (Actually, the integrability is retained if the constraints are replaced by $F_{jj} = c$, $j = 1, \cdots, \ell$, where c is a constant.)

4.1.2 Lax pair in vector representation

The Lax pair of the spinless $A_{\ell-1}$ model in the vector representation of $sl(\ell)$ can be readily extended to the spin generalization as follows:

$$\begin{aligned} L(z) &= \sum_{j=1}^{\ell} p_j E_{jj} + \sum_{j \neq k} \sigma(q_j - q_k, z) F_{kj} E_{jk}, \\ M(z) &= -\sum_{j \neq k} \sigma(q_j - q_k, z)(\rho(q_j - q_k) + \rho(z - q_j + w_k)) F_{kj} E_{jk}, \end{aligned} \tag{4.5}$$

where

$$\rho(u) = \frac{\theta_1'(u)}{\theta_1(u)}, \quad \sigma(u, z) = \frac{\theta_1(u - z)\theta_1'(0)}{\theta_1(z)\theta_1(u)}. \tag{4.6}$$

It is these functions that Felder and Wieczerkowski used in the KZB equation [17]. The function $\rho(u)$ is already familiar to us. The function $\sigma(u, z)$ is also just a disguise of the function $x(u, z)$ that we have used in the preceding sections:

$$\sigma(u, z) = -x(u, z). \tag{4.7}$$

We however dare to retain the notation of Felder and Wieczerkowski so as to stress the similarity with their work. In these notations, the aforementioned functional identities of $x(u, z)$ and $y(u, z)$ can be rewritten

$$\sigma(u, z)\sigma(v, z)(\rho(v) + \rho(z - v) - \rho(u) - \rho(z - u)) = \sigma(u + v, z)(\wp(u) - \wp(v)), \tag{4.8}$$

$$2\sigma(u, z)\sigma(-u, z)(\rho(u) + \rho(z - u)) = -\wp'(u), \tag{4.9}$$

$$\sigma(u, z)\sigma(-u, z) = \wp(z) - \wp(u). \tag{4.10}$$

Using these functional identities, one can derive the Lax equation

$$\frac{\partial L(z)}{\partial t} = [L(z), M(z)]. \tag{4.11}$$

Note that the constraints (4.4) are always assumed when we consider the Lax equation. Thus the spin generalization, too, is an isospectral integrable system. An involutive set of conserved quantities obtained from the traces $\mathrm{Tr}\, L(z)^k$, $k = 2, 3, , \cdots$. The Hamiltonian itself can be reproduced from the quadratic trace.

The matrix $F = \sum_{j \neq k} F_{kj} E_{jk}$, which is the residue of $L(z)$ at $z = 0$, stays on a coadjoint orbit of $sl(\ell)$ as t varies. The phase space of the spin generalization can be

thereby restricted to the direct product of the phase space of Calogero-Moser particles and a coadjoint orbit of various dimensions in the dual space of $sl(\ell)$. The lowest dimensional non-trivial coadjoint orbit can be parametrized by 2ℓ variables a_j, b_j $(j = 1, \cdots, \ell)$ as

$$F_{jk} = igb_j a_k \quad (j \neq k), \tag{4.12}$$

where g is a constant. These reduced spin degrees of freedom, however, can be eliminated by a diagonal gauge transformation of the Lax equations. (This does not mean that a_j and b_j are non-dynamical. The elimination procedure is done by partially solving the equations of motion for those variables.) This gauge transformation in turn gives rise to non-zero diagonal elements in $M(z)$, and the outcome is nothing but the Lax equation of the spinless elliptic Calogero-Moser system with coupling constant g. The spinless system is thus embedded in the spin generalization.

4.1.3 Isomonodromic system

There is no substantial difference in the construction of an isomonodromic system. The equations of motion are given by

$$\begin{aligned} 2\pi i \frac{dq_j}{d\tau} &= p_j, \quad 2\pi i \frac{dp_j}{d\tau} = \sum_{k \neq j} \wp'(q_j - q_k) F_{jk} F_{kj}, \\ 2\pi i \frac{dF_{jk}}{d\tau} &= \sum_{m \neq j} \wp(q_j - q_m) F_{jm} - \sum_{m \neq k} \wp(q_m - q_k) F_{mk}. \end{aligned} \tag{4.13}$$

(Terms including F_{jj}'s have been eliminated by the constraints.) The Lax equation, too, can be written in the same form

$$2\pi i \frac{\partial L(z)}{\partial \tau} + \frac{\partial M(z)}{\partial z} = [L(z), M(z)]. \tag{4.14}$$

Behind this Lax equation is the "heat equation"

$$2\pi i \frac{\partial \sigma(u,z)}{\partial \tau} + \frac{\partial^2 \sigma(u,z)}{\partial u \partial z} = 0 \tag{4.15}$$

satisfied by $\sigma(u, z)$. The final piece of the ring is the monodromy of $L(z)$ and $M(z)$:

$$\begin{aligned} & L(z+1) = L(z), \quad M(z+1) = M(z), \\ & L(z+\tau) = e^{2\pi i Q} L(z) e^{-2\pi i Q}, \\ & M(z+\tau) = e^{2\pi i Q}(M(z) + 2\pi i L(z)) e^{-2\pi i Q} - 2\pi i P. \end{aligned} \tag{4.16}$$

As opposed to the root type Lax pairs, the ordinary differential equation

$$\frac{dY(z)}{dz} = L(z)Y(z) \tag{4.17}$$

on the torus E_τ has only one regular singularity at $z = 0$. Thus the local monodromy matrix Γ_0 and the global monodromy matrices Γ_α and Γ_β are all that are invariant under the deformations.

4.2 Preliminaries for general simple Lie algebra

Let $\mathfrak{g}$ be a (complex) simple Lie algebra of rank ℓ, $\mathfrak{h}$ a Cartan subalgebra, and Δ the associated root system. The Cartan subalgebra induces a root space decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha. \tag{4.18}$$

We choose a basis $\{e_\alpha, h_\mu \mid \alpha \in \Delta,\ \mu = 1, \cdots, \ell\}$ of $\mathfrak{g}$ as follows:

1. h_μ, $\mu = 1, \ldots, \ell$, are an orthonormal basis of $\mathfrak{h}$ with respect to the Killing form $B : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$, i.e.,

   $$B(h_\mu, h_\nu) = \delta_{\mu\nu}. \tag{4.19}$$

   The Killing form induces an isomorphism $\mathfrak{h}^* = Hom(\mathfrak{h}, \mathbb{C}) \simeq \mathfrak{h}$, which determines an element h_α for each $\alpha \in \mathfrak{h}^*$. In terms of the basis h_μ of $\mathfrak{h}$, this map can be written explicitly:

   $$\alpha \mapsto h_\alpha = \sum_{\mu=1}^{\ell} \alpha(h_\mu)h_\mu, \tag{4.20}$$

2. The root subspace $\mathfrak{g}_\alpha$ is one dimensional. e_α is a basis of $\mathfrak{g}_\alpha$ such that

   $$[e_\alpha, e_{-\alpha}] = h_\alpha. \tag{4.21}$$

   This choice of e_α amounts to the normalization

   $$B(e_\alpha, e_{-\alpha}) = 1. \tag{4.22}$$

The Lie brackets of the basis elements other than $[e_\alpha, e_{-\alpha}]$ now takes the form

$$\begin{aligned} [e_\alpha, e_\beta] &= N_{\alpha,\beta} e_{\alpha+\beta} \quad (\alpha+\beta \neq 0), \\ [h_\mu, e_\alpha] &= \alpha(h_\mu) e_\alpha, \\ [h_\mu, h_\nu] &= 0. \end{aligned} \tag{4.23}$$

The structure constants $N_{\alpha,\beta}$ are anti-symmetric with respect to the indices, and vanish if $\alpha + \beta \notin \Delta$. The following general relation among the structure constants will be used in the course of the proof of a Lax equation

Lemma 1

$$N_{-\beta,\alpha+\beta} = N_{-\alpha,-\beta} = N_{\alpha+\beta,-\alpha}. \tag{4.24}$$

Proof. If $\alpha = \beta$, this relation is trivially satisfied, because all the structure constants vanish. Let us consider the case where $\alpha \neq \beta$. By the Jacobi identity, we have

$$[e_{\alpha+\beta}, [e_{-\alpha}, e_{-\beta}]] = [[e_{\alpha+\beta}, e_{-\alpha}], e_{-\beta}] + [e_{-\alpha}, [e_{\alpha+\beta}, e_{-\beta}]].$$

This implies the identity

$$N_{-\alpha,-\beta} h_{\alpha+\beta} = N_{\alpha+\beta,-\alpha} h_\beta - N_{\alpha+\beta,-\beta} h_\alpha,$$

which, by the relation $h_{\alpha+\beta} = h_\alpha + h_\beta$, can be rewritten

$$(N_{-\alpha,-\beta} + N_{\alpha+\beta,-\beta}) h_\alpha + (N_{-\alpha,-\beta} - N_{\alpha+\beta,-\alpha}) h_\beta = 0.$$

Since we have assumed that $\alpha \neq \beta$, h_α and h_β are linearly independent, so that the two coefficients in this linear retion are equal to zero. *Q.E.D.*

We can now specify the classical spin variables for a general simple Lie algebra. Those spin variables, by definition, are coordinates of the dual space $\mathfrak{g}^* = Hom(\mathfrak{g}, \mathbb{C})$. Let F_α and G_μ be the coordinates dual to the above basis e_α and h_μ. In other words, they are the coefficients of e_α and h_μ in the linear combination

$$\sum_{\alpha\in\Delta} F_{-\alpha} e_\alpha + \sum_{\mu=1}^{\ell} G_\mu h_\mu \tag{4.25}$$

that realizes the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ induced by the Killing form. The Kostant-Kirillov Poisson structure on $\mathfrak{g}^*$ determine the Poisson brackets of these spin variables, which take the same form as the Lie brackets of the Lie algebra basis:

$$\begin{aligned}
\{F_\alpha, F_{-\alpha}\} &= G_\alpha = \sum_{\mu=1}^{\ell} \alpha(h_\mu) G_\mu, \\
\{F_\alpha, F_\beta\} &= N_{\alpha,\beta} F_{\alpha+\beta} \quad (\alpha + \beta \neq 0), \\
\{G_\mu, F_\alpha\} &= \alpha(h_\mu) F_\alpha, \\
\{G_\mu, G_\nu\} &= 0.
\end{aligned} \tag{4.26}$$

4.3 Spin generalization for general simple Lie algebra

4.3.1 Hamiltonian formalism

The spin generalization based on $\mathfrak{g}$, too, is a constrained Hamiltonian system defined on $\mathfrak{h} \times \mathfrak{h} \times \mathfrak{g}^*$ by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} B(p,p) - \frac{1}{2} \sum_{\alpha \in \Delta} \wp(\alpha(q)) F_{-\alpha} F_\alpha \tag{4.27}$$

and the constraints

$$G_\mu = 0 \quad (\mu = 1, \cdots, \ell). \tag{4.28}$$

Here q and p are understood to take values in $\mathfrak{h}$. $B(p,q)$ and $\alpha(q)$ amount to $p \cdot p$ and $\alpha \cdot q$ in the models based on root systems. Let us use the same "dot notation" for the Killing form $\mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ and the pairing $\mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$. The Hamiltonian then takes a more familiar form:

$$\mathcal{H} = \frac{1}{2} p \cdot p - \frac{1}{2} \sum_{\alpha \in \Delta} \wp(\alpha \cdot q) F_{-\alpha} F_\alpha \tag{4.29}$$

The equations of motion can be readily written down in the language of the coordinates $q_\mu = q \cdot h_\mu$ and momenta $p_\mu = p \cdot h_\mu$ of Calogero-Moser particles and the spin variables F_α and G_μ on $\mathfrak{g}^*$:

$$\begin{aligned}
\frac{dq_\mu}{dt} &= p_\mu, \\
\frac{dp_\mu}{dt} &= -\frac{1}{2} \sum_{\alpha \in \Delta} \alpha \cdot h_\mu \wp'(\alpha \cdot q) F_{-\alpha} F_\alpha,
\end{aligned}$$

$$\begin{aligned}
\frac{dF_\alpha}{dt} &= -\sum_{\beta\in\Delta,\alpha-\beta\in\Delta} \wp(\beta\cdot q)F_{\alpha-\beta}F_\beta N_{\alpha,-\beta} - \wp(\alpha\cdot q)G_\alpha F_\alpha, \\
\frac{dG_\mu}{dt} &= 0. \qquad (4.30)
\end{aligned}$$

In particular, the diagonal elements G_μ of the spin variables are conserved quantities. One can thereby safely put the aforementioned constraints.

4.3.2 Lax pair

The integrability of our spin generalization is ensured by the existence of a Lax pair as follows.

Proposition 6 *Let V be any finite dimensional representation of $\mathfrak{g}$, and E_α and H_μ the endomorphisms on V that represent e_α and h_μ. Then the endomorphisms*

$$\begin{aligned}
L(z) &= P + \sum_{\alpha\in\Delta}\sigma(\alpha\cdot q, z)F_{-\alpha}E_\alpha, \quad P = \sum_{\mu=1}^{\ell} p_\mu H_\mu, \\
M(z) &= -\sum_{\alpha\in\Delta}\sigma(\alpha\cdot q, z)(\rho(\alpha\cdot q) + \rho(z-\alpha\cdot q))F_{-\alpha}E_\alpha \qquad (4.31)
\end{aligned}$$

on V satisfy the Lax equation

$$\frac{\partial L(z)}{\partial t} = [L(z), M(z)]. \qquad (4.32)$$

Proof. Using the equations of motion and the constraints, one can express the t-derivative of the L-matrix as

$$\frac{\partial L(z)}{\partial t} = I + II + III, \qquad (4.33)$$

where

$$\begin{aligned}
I &= \sum_{\mu=1}^{\ell}\frac{dp_\mu}{dt}H_\mu = -\frac{1}{2}\sum_{\alpha\in\Delta}\wp'(\alpha\cdot q)F_{-\alpha}F_\alpha H_\alpha, \\
II &= \sum_{\alpha\in\Delta}\sum_{\mu=1}^{\ell}\frac{d\alpha\cdot q}{dt}\left.\frac{\partial\sigma(u,z)}{\partial u}\right|_{u=\alpha\cdot q}F_{-\alpha}E_\alpha \\
&= -\sum_{\alpha\in\Delta}\alpha\cdot\alpha\sigma(\alpha\cdot q, z)(\rho(\alpha\cdot q)+\rho(z-\alpha\cdot q))F_{-\alpha}E_\alpha, \\
III &= \sum_{\alpha\in\Delta}\sigma(\alpha\cdot q, z)\frac{dF_{-\alpha}}{dt}E_\alpha \\
&= -\sum_{\alpha,\beta\in\Delta,\alpha+\beta\neq 0}\sigma(\alpha\cdot q, z)\wp(\beta\cdot q)F_{-\alpha-\beta}F_\beta N_{-\alpha,-\beta}E_\alpha.
\end{aligned}$$

Similarly, the commutator of the Lax pair can be written

$$[L(z), M(z)] = IV + V + VI, \tag{4.34}$$

where VI stands for terms from the commutator $[P, M(z)]$,

$$\begin{aligned}
IV &= -\sum_{\alpha\in\Delta} \sigma(\alpha\cdot q, z)(\rho(\alpha\cdot q) + \rho(z-\alpha\cdot q))F_{-\alpha}[P, E_\alpha] \\
&= -\sum_{\alpha\in\Delta} \sigma(\alpha\cdot q, z)(\rho(\alpha\cdot q) + \rho(z-\alpha\cdot q))\alpha\cdot pF_{-\alpha}E_\alpha,
\end{aligned}$$

and $V + VI$ are the the other terms grouped into the Cartan part (V) and the off-Cartan part (VI),

$$\begin{aligned}
V &= -\sum_{\alpha\in\Delta} \sigma(-\alpha\cdot q, z)\sigma(\alpha\cdot q, z)(\rho(\alpha\cdot q) + \rho(z-\alpha\cdot q))F_{-\alpha}F_\alpha[E_\alpha, E_{-\alpha}] \\
&= -\sum_{\alpha\in\Delta} \sigma(-\alpha\cdot q, z)\sigma(\alpha\cdot q, z)(\rho(\alpha\cdot q) + \rho(z-\alpha\cdot q))F_{-\alpha}F_\alpha H_\alpha, \\
VI &= -\sum_{\alpha,\beta\in\Delta,\alpha+\beta\neq 0} \sigma(\alpha\cdot q, q)\sigma(\beta\cdot q, z)(\rho(\beta\cdot q) + \rho(z-\beta\cdot q))F_{-\alpha}F_\alpha[E_\alpha, E_\beta] \\
&= -\sum_{\alpha,\beta\in\Delta,\alpha+\beta\neq 0} \sigma(\alpha\cdot q, z)\sigma(\beta\cdot q, z)(\rho(\beta\cdot q) + \rho(z-\beta\cdot q))F_{-\alpha}F_{-\beta}N_{\alpha,\beta}E_{\alpha+\beta}.
\end{aligned}$$

It is obvious that $IV = II$. Using (4.9), we can readily see that $V = I$. Thus it remains to prove that $VI = III$. This is achieved as follows:

$$\begin{aligned}
VI &= -\frac{1}{2}\sum_{\alpha,\beta\in\Delta,\alpha+\beta\neq 0} \sigma(\alpha\cdot q, z)\sigma(\beta\cdot q, z)\Big(\rho(\beta\cdot q) + \rho(z-\beta\cdot q) \\
&\quad -\rho(z-\alpha\cdot q) - \rho(\alpha\cdot q)\Big)F_{-\alpha}F_{-\beta}N_{\alpha,\beta}E_{\alpha+\beta} \\
&\quad \text{[symmetrized with respect to } \alpha \text{ and } \beta] \\
&= -\frac{1}{2}\sum_{\alpha,\beta\in\Delta,\alpha+\beta\neq 0} \sigma((\alpha+\beta)\cdot q, z)(\wp(\alpha\cdot q) - \wp(\beta\cdot q))F_{-\alpha}F_{-\beta}N_{\alpha,\beta}E_{\alpha+\beta}. \\
&\quad \text{[(4.8) is used]} \\
&= \sum_{\alpha,\beta\in\Delta,\alpha+\beta\neq 0} \sigma((\alpha+\beta)\cdot q, z)\wp(\beta\cdot q)F_{-\alpha}F_{-\beta}N_{\alpha,\beta}E_{\alpha+\beta} \\
&\quad \text{[asymmetrized with respect to } \alpha \text{ and } \beta] \\
&= \sum_{\alpha,\beta\in\Delta,\alpha+\beta\neq 0} \sigma(\alpha\cdot q, z)\wp(\beta\cdot q)F_{-\alpha,-\beta}F_\beta N_{\alpha+\beta,-\beta}E_\alpha. \\
&\quad \text{[substituting } \beta\to-\beta \text{ and } \alpha\to\alpha+\beta]
\end{aligned}$$

Finally using the identity $N_{\alpha+\beta,-\beta} = -N_{-\alpha,-\beta}$, cf. (4.24), we find that the last sum is equal to III. *Q.E.D.*

Note that the above proof persists to be meaningful if E_α and H_μ are replaced by the Lie algebra elements e_α and h_μ. In other words, the Lax equation actually lives in the Lie algebra $\mathfrak{g}$ itself rather than its representations. This resembles the case of the Toda systems.

4.3.3 Isomonodromic System

The passage to an isomonodromic analogue is straightforward. Replacing $d/dt \to 2\pi i d/d\tau$, one obtains the non-autonomous system

$$\begin{aligned} 2\pi i \frac{dq_\mu}{d\tau} &= p_\mu, \\ 2\pi i \frac{dp_\mu}{d\tau} &= -\frac{1}{2}\sum_{\alpha\in\Delta} \alpha \cdot h_\mu \wp'(\alpha \cdot q) F_{-\alpha} F_\alpha, \\ 2\pi i \frac{dF_\alpha}{d\tau} &= -\sum_{\beta\in\Delta, \alpha-\beta\in\Delta} \wp(\beta \cdot q) F_{\alpha-\beta} F_\beta N_{\alpha,-\beta}. \end{aligned} \tag{4.35}$$

(Terms icluding G_μ's have been eliminated by the constraints.) These equations can be converted to the Lax equation

$$2\pi i \frac{\partial L(z)}{\partial \tau} + \frac{\partial M(z)}{\partial z} = [L(z), M(z)]. \tag{4.36}$$

The monodromy of $L(z)$ and $M(z)$, too, takes the same form:

$$\begin{aligned} & L(z+1) = L(z), \quad M(z+1) = M(z), \\ & L(z+\tau) = e^{2\pi i Q} L(z) e^{-2\pi i Q}, \\ & M(z+\tau) = e^{2\pi i Q}(M(z) + 2\pi i L(z)) e^{-2\pi i Q} - 2\pi i P, \end{aligned} \tag{4.37}$$

where $Q = \sum_{\mu=1}^{\ell} q_\mu H_\mu$. The Lax equation implies that the monodromy data of the ordinary differential equation

$$\frac{dY(z)}{dz} = L(z) Y(z) \tag{4.38}$$

on the torus E_τ is invariant as τ varies. $Y(z)$ now take values in the representation space V; the monodromy around a singular point or of a cycle of E_τ is represented by a

linear transformation on V. The ordinary differential equation has a regular singularity at $z = 0$ only. The local monodromy around this singular point is a linear transformation $\Gamma_0 \in GL(V)$. Similarly, the global monodromy along the α and β cycles give $\Gamma_\alpha, \Gamma_\beta \in GL(V)$. These linear transformations Γ_0, Γ_α and Γ_β are the monodromy data that are left invariant.

5 Conclusion

We have thus demonstrated that various models of the elliptic Calogero-Moser systems are accompanied with an isomonodromic partner. A technical clue is the choice of fundamental functions $x(u, z)$, $y(u, z)$, etc. in the Lax pair $L(z)$ and $M(z)$. For $L(z)$ and $M(z)$ to give an isomonodromic Lax pair, these functions are required to satisfy a kind of "heat equation" besides the functional equations. We have illustrated the construction of the isomonodromic Lax pair for several typical cases — the Lax pair of the $A_{\ell-1}$ mode in the vector representation, the root type Lax pair for various untwisted and twisted models, and the Lax pair of the spin generalizations.

The most interesting case in the context of Manin's equation is the root type Lax pair for the extended twisted BC_ℓ model (or, equivalently, the Inozemtsev system). The root type Lax pair based on short roots of the BC_ℓ root system consists of $2\ell \times 2\ell$ matrices.

The construction of a Lax pair, however, is merely the first step towards a full understanding of Manin's equation and its possible generalizations. The next isse is to elucidate the meaning of the affine Weyl group symmetries, various special solutions, etc. in this framework. Recent works by Noumi and Yamada [20], Deift, Its, Kapaev and Zhou [21] and Kitaev and Korotkin [22] are very suggestive in this respect.

The spin generalization that we have discussed is a special case of a more general multi-spin system, i.e., the elliptic Calogero-Moser systems coupled to "Gaudin spins" sitting at the punctures of a punctured torus [9, 10]. This is the Hitchin system on a punctured torus; we have considered the case with only one puncture located at $z = 0$. It is rather straightforward, though more complicated, to generalize our Lax pair to the multi-spin generalization. This gives a generalization, to other simple Lie groups, of the $SU(2)$ isomonodromic system of Korotkin and Samtleben [23]. The dynamical r-matrix in the work of Felder and Wieczerkowski [17] plays a central role here. We shall report

this result elsewhere.

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A Proof of Functional Identities and Heat Equation for Untwisted Models

A.1 Proof of (2.8)

Let $f(u,v,z)$ denote the difference of both hand sides of (2.8):

$$f(u,v,z) = x(u,z)y(v,z) - y(u,z)x(v,z) - x(u+v,z)(\wp(u) - \wp(v)). \tag{A.1}$$

This function turns out to have the following analytical properties:

1. $f(u,v,z)$ has the same quasi-periodicity as $x(u,z)$ on the u plane, i.e.,

$$f(u+1,v,z) = f(u,v,z), \quad f(u+\tau,v,z) = e^{2\pi i z} f(u,v,z). \tag{A.2}$$

2. $f(u,v,z)$ is an entire function on the u plane.

The first property is obvious from the quasi-periodicity of $x(u,z)$ and the periodicity of $\wp(u)$. Furthermore, poles of $f(u,v,z)$ can appear only at the lattice points $u = m + n\tau$ ($m,n \in \mathbb{Z}$) on the u plane. Therefore, in order to verify the second property, we have only to show that $f(u,v,z)$ is non-singular at these points. Actually, because of the quasi-periodicity, it is sufficient to consider the point $u = 0$ only. As $u \to 0$, the singular terms

$x(u,z)$, $y(u,z)$ and $\wp(u)$ in $f(u,v,z)$ behave as

$$\begin{aligned} x(u,z) &= \frac{1}{u} + O(1), \\ y(u,z) &= -\frac{1}{u^2} + O(1), \\ \wp(u) &= \frac{1}{u^2} + O(u^2) \end{aligned} \tag{A.3}$$

so that

$$\begin{aligned} f(u,v,z) &= \left(\frac{1}{u} + O(1)\right) y(v,z) - \left(-\frac{1}{u^2} + O(1)\right) x(v,z) \\ &\quad - \left(x(u,z) + y(u,z)u + O(u^2)\right)\left(\frac{1}{u^2} - \wp(v) + O(u^2)\right) \\ &= O(1). \end{aligned} \tag{A.4}$$

We can thus verify the above two properties of $f(u,v,z)$.

Actually, any function with these two properties should vanish identically. This can be seen in several different ways. The shortest will be to resort to algebraic geometry of line bundles on the torus E_τ. A more elementary proof is to consider the quotient $f(u,v,z)/x(u,z)$. This quotient is a doubly-periodic meromorphic function, and all possible poles are located at the lattice points $u = m + n\tau$ $(m, n \in \mathbb{Z})$, and at most of first order. In other words, $f(u,v,z)/x(u,z)$ is a meromorphic function on the torus with the only possible pole at $u = 0$, but the order of pole cannot be greater than one. Such a function has to be a constant. On the other hand, because of the pole of $x(u,z)$ at $u = 0$, $f(u,v,z)/x(u,z)$ has a zero at $u = 0$. Therefore the constant should be equal to zero.

A.2 Proof of (2.9) and (2.10)

(2.9) can be readily derived from (2.8) by letting $v \to -u$. Let us consider (2.10). By (2.9),

$$\frac{\partial}{\partial u}\Big(x(u,z)x(-u,z)\Big) = -x(u,z)y(-u,z) + y(u,z)x(-u,z) = -\wp'(u). \tag{A.5}$$

Consequently,

$$x(u,z)x(-u,z) = -\wp(u) + (\text{independent of } u). \tag{A.6}$$

Since $x(u,z) = -x(z,u) = -x(-u,-z)$, the left hand side of the last relation is in fact an anti-symmetric function of u and z. Therefore,

$$x(u,z)x(-u,z) = \wp(z) - \wp(u) + \text{const.} \tag{A.7}$$

Now consider the limit as $u \to z$. Both $x(u,z)x(-u,z)$ and $\wp(z) - \wp(u)$ tend to zero in this limit. Thus the constant on the right hand side has to be zero.

A.3 Proof of (2.18)

Let us rewrite the both hand sides of (2.18) into a more accessible form. Differentiating $x(u,z)$ by τ gives

$$\frac{\partial x(u,z)}{\partial \tau} = x(u,z)\frac{\partial}{\partial \tau}\Big(\log\theta_1(z-u) + \log\theta_1'(0) - \log\theta_1(z) - \log\theta_1(u)\Big). \tag{A.8}$$

By the heat equation (2.24) of the Jacobi theta function,

$$4\pi i\frac{\partial}{\partial \tau}\theta_1(u) = \frac{\theta_1''(u)}{\theta_1(u)} = \frac{\partial}{\partial u}\left(\frac{\theta_1'(u)}{\theta_1(u)}\right) + \left(\frac{\theta_1'(u)}{\theta_1(u)}\right)^2 = \rho'(u) + \rho(u)^2. \tag{A.9}$$

Letting $u \to 0$ and recalling the singular behavior of $\rho(u)$ at $u = 0$, we obtain

$$4\pi i\frac{\partial}{\partial \tau}\log\theta_1'(0) = \lim_{u\to 0}(\rho'(u) + \rho(u)) = \frac{\theta_1'''(0)}{\theta_1'(0)}. \tag{A.10}$$

Plugging these formulae into the above expression of $\partial x(u,z)/\tau$ gives

$$4\pi i\frac{\partial x(u,z)}{\partial \tau} = x(u,z)f(u,z), \tag{A.11}$$

where

$$f(u,z) = \rho'(z-u) + \rho(z-u)^2 + \frac{\theta_1'''(0)}{\theta_1'(0)} - \rho'(z) - \rho(z)^2 - \rho'(u) - \rho(u)^2. \tag{A.12}$$

On the other hand, we have

$$\frac{\partial x(u,z)}{\partial u\partial z} = -\frac{\partial}{\partial z}\Big(x(u,z)(\rho(u) + \rho(z-u))\Big) = -x(u,z)g(u,z), \tag{A.13}$$

where

$$g(u,z) = (\rho(z-u) - \rho(z))(\rho(u) + \rho(z-u)) + \rho'(z-u). \tag{A.14}$$

The goal is to verify that $f(u,z) = 2g(u,z)$. It is sufficient to prove the following two properties of $f(u,z) - 2g(u,z)$, because such a function has to be identically zero.

1. $f(u,z)-2g(u,z)$ is a doubly-periodic function on the u plane with primitive periods 1 and τ.

2. $f(u,z)-2g(u,z)$ is an entire function, and has a zero at $u=0$.

The first property is obvious if one notices the following quasi-periodicity of $f(u,z)$ and $g(u,z)$:

$$
\begin{aligned}
f(u+1,z) &= f(u,z), \qquad f(u+\tau,z) = f(u,z) + 4\pi i(\rho(u)+\rho(z-u)), \\
g(u+1,z) &= g(u,z), \qquad g(u+\tau,z) = g(u,z) + 2\pi i(\rho(u)+\rho(z-u)).
\end{aligned} \tag{A.15}
$$

Let us check the second property. Possible poles of $f(u,z)$ and $g(u,z)$ are located at the two points $u=0$ and $u=z$ of the fundamental domain of the period lattice $\mathbb{Z}+\tau\mathbb{Z}$. Again recalling the singular behavior of $\rho(u)$ at $u=0$, one can confirm by straightforward calculations that

$$
f(u,z) = O(u), \quad g(u,z) = O(u) \quad (u \to 0). \tag{A.16}
$$

Thus $f(u,z)-2g(u,z)$ turns out to be non-singular and have a zero at $u=0$. Similarly, one can see that $f(u,z)-2g(u,z)$ is non-singular at $u=z$.

B Verification of Lax Pair for Extended Twisted BC_ℓ Model

To prove the Lax equation, it is sufficient to derive the following three equations:

$$
\begin{aligned}
\frac{\partial X_a(z)}{\partial t} &= [P, X_a(z)] \quad (a=1,2,3), && \text{(B.1)} \\
\frac{dp\cdot\mu}{dt} &= [X_1(z)+X_2(z)+X_3(z), Y_1(z)+Y_2(z)+Y_3(z)]_{\mu\mu}, && \text{(B.2)} \\
0 &= [X_1(z)+X_2(z)+X_3(z), D+Y_1(z)+Y_2(z)+Y_3(z)]_{\mu\nu} \quad (\mu\neq\nu). && \text{(B.3)}
\end{aligned}
$$

μ and ν run over the set Δ_s of short roots.

The proof of (B.1) is quite easy. Let us consider the case of $a=1$. The t-derivative of $X_1(z)$ can be written

$$
\frac{\partial X_1(z)}{\partial t} = i g_m \sum_{\alpha\in\Delta_m} \alpha\cdot p y(\alpha\cdot q, z) E(\alpha). \tag{B.4}
$$

Using the commutation relation $[P, E(\alpha)] = \alpha \cdot pE(\alpha)$, one can readily see that the right hand side is equal to $[P, X_1(z)]$. The other two in (B.1) can be similarly derived.

The rest of this appendix is devoted to the other two equations (B.2) and (B.3).

B.1 Proof of (B.2)

We calculate the diagonal elements

$$[X_a(z), Y_b(z)]_{\mu\mu} = \sum_{\nu\in\Delta_s} \Big(X_{a,\mu\nu}(z) Y_{b,\nu\mu}(z) - Y_{b,\mu\nu}(z) X_{a,\nu\mu}(z) \Big) \tag{B.5}$$

of the nine commutators one-by-one.

B.1.1 Vanishing terms

Some part of the matrix elements of $X_a(z)$ and $Y_b(z)$ turn out to vanish by the nature of the BC_ℓ root system:

$$X_{1,\mu,-\mu}(z) = Y_{1,\mu,-\mu}(z) = 0, \tag{B.6}$$

$$X_{2,\mu\nu}(z) = Y_{2,\mu\nu}(z) = 0 \quad (\mu \neq -\nu), \tag{B.7}$$

$$X_{3,\mu\nu}(z) = Y_{3,\mu\nu}(z) = 0 \quad (\mu \neq -\nu). \tag{B.8}$$

The first relation is due to the fact that $\mu - (-\mu) = 2\mu$ can never be a middle root. The second and third relations are obvious if one notices that $\mu - \nu$ is a long root (or, equivalently, twice a short root) if and only if $\mu = -\nu$.

In particular,

$$[X_1(z), Y_2(z)]_{\mu\mu} = [X_1(z), Y_3(z)]_{\mu\mu} = [X_2(z), Y_1(z)]_{\mu\mu} = [X_3(z), Y_1(z)]_{\mu\mu} = 0. \tag{B.9}$$

B.1.2 Calculation of $[X_1(z), Y_1(z)]_{\mu\mu}$

By definition,

$$\begin{aligned}[X_1(z), Y_1(z)]_{\mu\mu} = -g_m^2 \sum_{\nu\in\Delta_s, \mu-\nu\in\Delta_m} \Big(& x((\mu-\nu)\cdot q, z) y((\nu-\mu)\cdot q, z) \\ & - y((\mu-\nu)\cdot q, z) x((\nu-\mu)\cdot q, z) \Big). \end{aligned} \tag{B.10}$$

We rewrite this sum to a sum over the middle root $\alpha = \mu - \nu$. Since the middle roots α of this form are characterized by the condition that $\alpha \cdot \mu = 1$, the right hand side can be rewritten

$$-g_m^2 \sum_{\alpha \in \Delta_m, \alpha \cdot \mu = 1} \Big(x(\alpha \cdot q, z) y(-\alpha \cdot q, z) - y(\alpha \cdot q, z) x(-\alpha \cdot q, z) \Big).$$

Actually, the possible values of $\alpha \cdot \mu$ are limited to 0 and ± 1 only. Therefore this sum is equal to

$$-\frac{g_m^2}{2} \sum_{\alpha \in \Delta_m} \alpha \cdot \mu \Big(x(\alpha \cdot q, z) y(-\alpha \cdot q, z) - y(\alpha \cdot q, z) x(-\alpha \cdot q, z) \Big).$$

(The factor 1/2 compensates the contributions from $\alpha \cdot \mu = 1$ and $\alpha \cdot \mu = -1$.) Noting that $\alpha \cdot \mu = \{p \cdot \mu, \alpha \cdot q\}$, we can express $[X_1(z), Y_1(z)]$ as a Poisson bracket of the form

$$[X_1(z), Y_1(z)]_{\mu\mu} = \{p \cdot \mu, V_{11}\}, \tag{B.11}$$

where

$$V_{11} = \frac{g_m^2}{2} \sum_{\alpha \in \Delta_m} x(\alpha \cdot q, z) x(-\alpha \cdot q, z). \tag{B.12}$$

B.1.3 Contributions of other commutators

By (B.7) and (B.8), the diagonal elements of the other commutators are a sum of just two terms:

$$[X_a(z), Y_b(z)]_{\mu\mu} = X_{a,\mu,-\mu} Y_{b,-\mu,\mu} - Y_{b,\mu,-\mu} X_{a,-\mu,\mu}. \tag{B.13}$$

Let us consider the case of $a = 2$ and $b = 2$ in some detail. By definition,

$$\begin{aligned} & [X_2(z), Y_2(z)]_{\mu\mu} \\ = & -\Big(g_{l1} x(2\mu \cdot q, z) + g_{l2} x^{(2)}(2\mu \cdot q, z) \Big) \Big(g_{l1} y(-2\mu \cdot q, z) + g_{l2} y^{(2)}(-2\mu \cdot q, z) \Big) \\ & + \Big(g_{l1} y(2\mu \cdot q, z) + g_{l2} y^{(2)}(2\mu \cdot q, z) \Big) \Big(g_{l1} x(-2\mu \cdot q, z) + g_{l2} x^{(2)}(-2\mu \cdot q, z) \Big). \end{aligned}$$

Since $\alpha = 2\mu$ is a long root, and long roots with non-vanishing inner product with μ are 2μ and -2μ only, the right hand side can be rewritten

$$\begin{aligned} & -\frac{1}{4} \sum_{\alpha \in \Delta_l} \alpha \cdot \mu \Big(g_{l1} x(\alpha \cdot q, z) + g_{l2} x^{(2)}(\alpha \cdot q, z) \Big) \Big(g_{l1} y(-\alpha \cdot q, z) + g_{l2} y^{(2)}(-\alpha \cdot q, z) \Big) \\ & +\frac{1}{4} \sum_{\alpha \in \Delta_l} \alpha \cdot \mu \Big(g_{l1} y(\alpha \cdot q, z) + g_{l2} y^{(2)}(\alpha \cdot q, z) \Big) \Big(g_{l1} x(-\alpha \cdot q, z) + g_{l2} x^{(2)}(-\alpha \cdot q, z) \Big). \end{aligned}$$

(The factor 1/4 compensates the contributions from $\alpha \cdot \mu = 2$ and $\alpha \cdot \mu = -2$.) We can again cast this into a Poisson bracket:

$$[X_2(z), Y_2(z)]_{\mu\mu} = \{p \cdot \mu, V_{22}\}, \tag{B.14}$$

where

$$V_{22} = \frac{1}{4} \sum_{\alpha \in \Delta_l} \Big(g_{l1} x(\alpha \cdot q, z) + g_{l2} x^{(2)}(\alpha \cdot q, z) \Big) \Big(g_{l1} x(-\alpha \cdot q, z) + g_{l2} x^{(2)}(-\alpha \cdot q, z) \Big). \tag{B.15}$$

Similarly, one can obtain

$$\begin{aligned} &[X_2(z), Y_3(z)]_{\mu\mu} = \{p \cdot \mu, V_{23}\}, \quad [X_3(z), Y_2(z)]_{\mu\mu} = \{p \cdot \mu, V_{32}\}, \\ &[X_3(z), Y_3(z)]_{\mu\mu} = \{p \cdot \mu, V_{33}\}, \end{aligned} \tag{B.16}$$

where

$$\begin{aligned} V_{23} &= \frac{1}{2} \sum_{\alpha \in \Delta_s} \Big(g_{l1} x(2\alpha \cdot q, z) + g_{l2} x^{(2)}(2\alpha \cdot q, z) \Big) \Big(g_{s1} x(-\alpha \cdot q, 2z) + g_{s2} x^{(1/2)}(-\alpha \cdot q, 2z) \Big), \\ V_{32} &= \frac{1}{2} \sum_{\alpha \in \Delta_s} \Big(g_{s1} x(\alpha \cdot q, 2z) + g_{s2} x^{(1/2)}(\alpha \cdot q, 2z) \Big) \Big(g_{l1} x(-2\alpha \cdot q, z) + g_{l2} x^{(2)}(-2\alpha \cdot q, z) \Big), \\ V_{33} &= \sum_{\alpha \in \Delta_s} \Big(g_{s1} x(\alpha \cdot q, 2z) + g_{s2} x^{(1/2)}(\alpha \cdot q, 2z) \Big) \Big(g_{s1} x(-\alpha \cdot q, 2z) + g_{s2} x^{(1/2)}(-\alpha \cdot q, 2z) \Big). \end{aligned} \tag{B.17}$$

Collecting the results of these calculations, we find that the right hand side of (B.2) takes the form of the Poisson bracket $\{p \cdot \mu, V\}$, where

$$V = V_{11} + V_{22} + V_{23} + V_{32} + V_{33}. \tag{B.18}$$

B.1.4 Writing V in terms of $\wp$ functions

The final step is to rewrite V in terms of the Weierstrass $\wp$ functions. For V_{11}, this can be done by use of (2.10). The other parts are due to the following functional identities:

$$x^{(1/2)}(u, z) x^{(1/2)}(-u, z) = -\wp^{(1/2)}(u) + \wp^{(1/2)}(\frac{z}{2}), \tag{B.19}$$

$$x^{(2)}(u, z) x^{(2)}(-u, z) = -\wp^{(2)}(u) + \wp^{(2)}(2z), \tag{B.20}$$

$$x(u,2z)x^{(1/2)}(-u,2z)+x^{(1/2)}(u,2z)x(-u,2z)=-2\wp(u)+\text{const.}, \tag{B.21}$$

$$x(u,2z)x(-2u,z)+x(2u,z)x(-u,2z)=-\wp(u)+\text{const.}, \tag{B.22}$$

$$x(u,2z)x^{(2)}(-2u,z)+x^{(2)}(2u,z)x(-u,2z)=-\wp(u)+\text{const.}, \tag{B.23}$$

$$x^{(1/2)}(u,2z)x(-2u,z)+x(2u,z)x^{(1/2)}(-u,2z)=-\wp^{(1/2)}(u)+\text{const.}, \tag{B.24}$$

$$x^{(1/2)}(u,2z)x^{(2)}(-2u,z)+x^{(2)}(2u,z)x^{(1/2)}(-u,2z)]=-\wp(u)+\text{const.}, \tag{B.25}$$

$$x(u,z)x^{(2)}(-u,z)+x^{(2)}(u,z)x(-u,z)=-2\wp^{(2)}(u)+\text{const.} \tag{B.26}$$

The first two are substantially the same as (2.10) except that the variables and the primitive periods are rescaled. "const." in the other identities stand for terms that are independent of u, thereby negligible in the Poisson bracket with $p\cdot\mu$; remember that they are not absolute constants, but functions of z and τ. We shall prove these identities in Appendix C. Using these functional identities, one can see that V is equal to the potential part of the Hamiltonian $\mathcal{H}$, up to non-dynamical terms independent of p and q.

To summarize, we have shown that the sum of the (μ,μ) elements of the nine commutators coincides with the Poisson bracket $\{p\cdot\mu, V\}$, which is equal to $dp\cdot\mu/dt$ by the equations of motion of the model.

B.2 Proof of (B.3)

The proof can be separated into the cases where $\nu=-\mu$ and $\nu\neq\pm\mu$.

B.2.1 $\nu=-\mu$

The vanishing of the $(\mu,-\mu)$ elements of the commutators other than $[X_a(z),D]$ ($a=1,2,3$) and $[X_1(z),D]$ is immediate from (B.7) and (B.8). $[X_a(z),D]_{\mu,-\mu}$ vanishes because of the symmetry $D_{-\mu}=D_\mu$. As for $[X_1(z),Y_1(z)]_{\mu,-\mu}$, we have

$$\begin{aligned}[X_1(z),Y_1(z)]_{\mu,-\mu} &= -g_m^2\sum_{\nu\in\Delta_s\setminus\{\pm\mu\}} x((\mu-\nu)\cdot q,z)y((\nu+\mu)\cdot q,z)\\ &\quad +g_m^2\sum_{\nu\in\Delta_s\setminus\{\pm\mu\}} y((\mu-\nu)\cdot q,z)x((\nu+\mu)\cdot q,z).\end{aligned} \tag{B.27}$$

Bu substituting $\nu\to-\nu$, the second sum on the right hand turns out to be identical to the first sum. The two sums thus cancel with each other.

B.2.2 $\nu \neq \pm\mu$

The following can be readily seen by using (B.7) and (B.8):

$$
\begin{aligned}
&[X_2(z), D]_{\mu\nu} = [X_3(z), D]_{\mu\nu} = 0, \\
&[X_2(z), Y_2(z)]_{\mu\nu} = [X_2(z), Y_3(z)]_{\mu\nu} = [X_3(z), Y_3(z)]_{\mu\nu} = 0.
\end{aligned} \tag{B.28}
$$

The (μ, ν) elements of other commutators can be calculated as follows:

$$
\begin{aligned}
[X_1(z), D]_{\mu\nu} &= -X_{1,\mu\mu}(z)(D_\mu - D_\nu) \\
&= g_m x((\mu - \nu) \cdot q, z) \\
&\quad \times \Big(g_{s1}\wp(\mu \cdot q) + g_{s2}\wp^{(1/2)}(\mu \cdot q) + g_{l1}\wp(2\mu \cdot q) + g_{l2}\wp^{(2)}(2\mu \cdot q) \\
&\quad - g_{s1}\wp(\nu \cdot q) - g_{s2}\wp^{(1/2)}(\nu \cdot q) - g_{l1}\wp(2\nu \cdot q) - g_{l2}\wp^{(2)}(2\nu \cdot q) \\
&\quad + \sum_{\lambda \in \Delta_m, \alpha\cdot\mu = 1} \wp(\alpha \cdot q) - \sum_{\alpha \in \Delta_m, \alpha\cdot\nu = 1} \wp(\alpha \cdot q) \Big).
\end{aligned} \tag{B.29}
$$

$$
\begin{aligned}
[X_1(z), Y_1(z)]_{\mu\nu} &= \sum_{\lambda \in \Delta_s} \Big(X_{1,\mu\lambda}(z) Y_{1,\lambda\nu}(z) - Y_{1,\mu\lambda}(z) X_{1,\lambda\nu}(z) \Big) \\
&= -g_m^2 \sum_{\lambda \in \Delta_s \setminus \{\mu,\nu\}} \Big(x((\mu - \lambda) \cdot q, z) y((\lambda - \nu) \cdot q, z) \\
&\quad - y((\mu - \lambda) \cdot q, z) x((\lambda - \nu) \cdot q, z) \Big).
\end{aligned} \tag{B.30}
$$

$$
\begin{aligned}
[X_1(z), Y_2(z)]_{\mu\nu} &= X_{1,\mu,-\nu}(z) Y_{2,-\nu,\nu}(z) - Y_{2,\mu,-\mu}(z) X_{1,-\mu,\nu}(z) \\
&= -g_m x((\mu + \nu) \cdot q, z) \Big(g_{l1} y(-2\nu \cdot q, z) + g_{l2} y^{(2)}(-2\nu \cdot q, z) \Big) \\
&\quad + \Big(g_{l1} y(2\mu \cdot q, z) + g_{l2} y^{(2)}(2\mu \cdot q, z) \Big) g_m x(-(\mu + \nu) \cdot q, z).
\end{aligned} \tag{B.31}
$$

$$
\begin{aligned}
[X_1(z), Y_3(z)]_{\mu\nu} &= X_{1,\mu,-\nu}(z) Y_{3,-\nu,\nu}(z) - Y_{3,\mu,-\mu}(z) X_{1,-\mu,\nu}(z) \\
&= -g_m x((\mu + \nu) \cdot q, z) \Big(g_{s1} y(-\nu \cdot q, 2z) + g_{s2} y^{(1/2)}(-\nu \cdot q, 2z) \Big) \\
&\quad + \Big(g_{s1} y(\mu \cdot q, 2z) + g_{s2} y^{(1/2)}(\mu \cdot q, 2z) \Big) g_m x(-(\mu + \nu) \cdot q, z).
\end{aligned} \tag{B.32}
$$

$$
\begin{aligned}
[X_2(z), Y_1(z)]_{\mu\nu} &= X_{2,\mu,-\mu}(z) Y_{1,-\mu,\nu}(z) - Y_{1,\mu,-\nu}(z) X_{2,-\nu,\nu}(z) \\
&= -\Big(g_{l1} x(2\mu \cdot q, z) + g_{l2} x^{(2)}(2\mu \cdot q, z) \Big) g_m y(-(\mu + \nu) \cdot q, z)
\end{aligned}
$$

$$+g_m y((\mu+\nu)\cdot q,z)\Big(g_{l1}x(-2\nu\cdot q,z)+g_{l2}x^{(2)}(-2\nu\cdot q,z)\Big). \tag{B.33}$$

$$\begin{aligned}
[X_3(z),Y_1(z)]_{\mu\nu} &= X_{3,\mu,-\nu}(z)Y_{1,-\nu,\nu}(z)-Y_{1,\nu,-\nu}(z)X_{3,-\nu,\nu}(z)\\
&= -2\Big(g_{s1}x(\mu\cdot q,2z)+g_{s2}x^{(1/2)}(\mu\cdot q,2z)\Big)g_m y(-(\mu+\nu)\cdot q,z)\\
&\quad +2g_m y((\mu+\nu)\cdot q,z)\Big(g_{s1}x(-\nu\cdot q,2z)+g_{s2}x^{(1/2)}(-\nu\cdot q,2z)\Big).
\end{aligned} \tag{B.34}$$

We now sum up all these quantities, regroup terms into those multiplied by the same monomial of coupling constants, and show the cancellation in each partial sum. There are six monomials of coupling constants that can occur — g_m^2, $g_m g_{l1}$, $g_m g_{l2}$, $g_m g_{s1}$ and $g_m g_{s2}$.

Let us consider the terms multiplied by g_m^2. This is a sum of the following two quantities:

$$\begin{aligned}
I &= x((\mu-\nu)\cdot q,z)\Big(\sum_{\alpha\in\Delta_m,\alpha\cdot\mu=1}\wp(\alpha\cdot q)-\sum_{\alpha\in\Delta_m,\alpha\cdot\nu=1}\wp(\alpha\cdot q)\Big)\\
II &= -\sum_{\lambda\in\Delta_s\setminus\{\mu,\nu\}}\Big(x((\mu-\lambda)\cdot q,z)y((\lambda-\nu)\cdot q,z)\\
&\quad -y((\mu-\lambda)\cdot q,z)x((\lambda-\nu)\cdot q,z)\Big).
\end{aligned}$$

By the functional identity (2.8), we can rewrite II into a sum over middle roots:

$$\begin{aligned}
II &= -\sum_{\lambda\in\Delta_s\setminus\{\mu,\nu\}}x((\mu-\nu)\cdot q,z)\Big(\wp((\mu-\lambda)\cdot q)-\wp((\nu-\lambda)\cdot q)\Big)\\
&= -x((\mu-\nu)\cdot q,z)\Big(\sum_{\alpha\in\Delta_m,\alpha\cdot\mu=1}\wp(\alpha\cdot q)-\sum_{\alpha\in\Delta_m,\alpha\cdot\nu=1}\wp(\alpha\cdot q)\Big).
\end{aligned}$$

Here the sum over λ has been converted to a sum over α by putting $\alpha=\mu-\lambda$ and $\alpha=\nu-\lambda$ in the two $\wp$ function in the first line. Note that μ, ν and λ are all orthogonal to each other. We thus find that $I+II=0$.

For the other partial sums, we use the following functional identities, which we shall prove in Appendix C:

$$\begin{aligned}
&x(2u,z)y(-u-v,z)-y(2u,z)x(-u-v,z)+x(u+v,z)y(-2v,z)\\
&-y(u+v,z)x(-2v,z)-x(u-v,z)(\wp(2u)-\wp(2v))=0,
\end{aligned} \tag{B.35}$$

$$x^{(2)}(2u,z)y(-u-v,z) - y^{(2)}(2u,z)x(-u-v,z) + x(u+v,z)y^{(2)}(-2v,z)$$
$$-y(u+v,z)x^{(2)}(-2v,z) - x(u-v,z)(\wp^{(2)}(2u) - \wp^{(2)}(2v)) = 0, \qquad \text{(B.36)}$$
$$2x(u,2z)y(-u-v,z) - y(u,2z)x(-u-v,z) + x(u+v,z)y(-v,2z)$$
$$-2y(u+v,z)x(-v,2z) - x(u-v,z)(\wp(u) - \wp(v)) = 0, \qquad \text{(B.37)}$$
$$2x^{(1/2)}(u,2z)y(-u-v,z) - y^{(1/2)}(u,2z)x(-u-v,z) + x(u+v,z)y^{(1/2)}(-v,2z)$$
$$-2y(u+v,z)x^{(1/2)}(-v,2z) - x(u-v,z)(\wp^{(1/2)}(u) - \wp^{(1/2)}(v)) = 0. \qquad \text{(B.38)}$$

By these functional identities, we can confirm that all the partial sums regrouped by $g_m g_{l1}$, $g_m g_{l2}$, $g_m g_{s1}$ and $g_m g_{s2}$, respectively, cancel out.

C Proof of Functional Identities for Twisted Models

We here prove the functional identities that we have encountered in Appendix B. Although the proof is optimized to our choice of $x(u,z)$, $x^{(1/2)}(u,z)$ and $x^{(2)}(u,z)$, the same method can in principle apply to other solutions of the functional equations, such as the functions used by D'Hoker and Phong [15] and Bordner and Sasaki [14].

C.1 Analytical properties of $x^{(1/2)}(u,z)$ and $x^{(2)}(u,z)$

The proof of the identities including $x^{(1/2)}(u,z)$ and $x^{(2)}(u,z)$, like the proof in Appendix A, is based on the analytical properties of those functions.

- $x^{(1/2)}(u,z)$ has the following analytical properties:

  1. $x^{(1/2)}(u,z)$ is a meromorphic function of u and z. The poles on the u plane and the z plane are located at the lattice points $u = m/2 + n\tau$ and $z = m + 2n\tau$ $(m, n \in \mathbb{Z})$.
  2. $x^{(1/2)}(u,z)$ has the following quasi-periodicity:

$$x^{(1/2)}(u+\frac{1}{2},z) = x^{(1/2)}(u,z), \qquad x^{(1/2)}(u+\tau,z) = e^{2\pi i z}x^{(1/2)}(u,z),$$
$$x^{(1/2)}(u,z+1) = x^{(1/2)}(u,z), \qquad x^{(1/2)}(u,z+2\tau) = e^{4\pi i z}x^{(1/2)}(u,z). \qquad \text{(C.1)}$$

3. At the origin of the u and z planes, this function exhibits the following singular behavior:

$$x^{(1/2)}(u,z) = \frac{1}{u} - 2\rho(z \mid 2\tau) + O(u) \quad (u \to 0),$$
$$x^{(1/2)}(u,z) = -\frac{2}{z} + 2\rho(2u \mid 2\tau) + O(z) \quad (z \to 0). \tag{C.2}$$

- $x^{(2)}(u,z)$ has the following analytical properties:

1. $x^{(2)}(u,z)$ is a meromorphic function of u and z. The poles on the u plane and the z plane are located at the lattice points $u = 2m + n\tau$ and $z = m + n\tau/2$ $(m,n \in \mathbb{Z})$.
2. $x^{(2)}(u,z)$ has the following quasi-periodicity:

$$x^{(2)}(u+2,z) = x^{(2)}(u,z), \qquad x^{(2)}(u+\tau,z) = e^{2\pi iz}x^{(2)}(u,z),$$
$$x^{(2)}(u,z+1) = x^{(2)}(u,z), \qquad x^{(2)}(u,z+\frac{\tau}{2}) = e^{\pi iu}x^{(2)}(u,z). \tag{C.3}$$

3. At the origin of the u and z planes, this function exhibits the following singular behavior:

$$x^{(2)}(u,z) = \frac{1}{u} - \frac{1}{2}\rho(z \mid \frac{\tau}{2}) + O(u) \quad (u \to 0),$$
$$x^{(2)}(u,z) = -\frac{1}{2z} + \frac{1}{2}\rho(\frac{u}{2} \mid \frac{\tau}{2}) + O(z) \quad (z \to 0). \tag{C.4}$$

C.2 Proof of (B.35) – (B.38)

These four identities can be treated in much the same way. Let us illustrate the proof for (B.35) only. Since the line of the proof is almost the same as the proof of (2.8), we show an outline of the proof and leave the details to the reader.

Let $f(u,v,z)$ denote the left hand side of (B.35):

$$\begin{aligned} f(u,v,z) &= x(2u,z)y(-u-v,z) - y(2u,z)x(-u-v,z) + x(u+v,z)y(-2v,z) \\ &\quad -y(u+v,z)x(-2v,z) - x(u-v,z)(\wp(2u) - \wp(2v)). \end{aligned} \tag{C.5}$$

Our task is to show the following analytic properties of $f(u,v,z)$, which imply that this function is identically zero:

1. $f(u,v,z)$ has the quasi-periodicity as follows:

$$f(u+1,v,z) = f(u,v,z), \quad f(u+\tau,v,z) = e^{2\pi iz} f(u,v,z). \tag{C.6}$$

2. $f(u,v,z)$ is an entire function on the u plane.

The first property is immediate from the quasi-periodicity of $x(u,z)$, etc. Furthermore, it is obvious from the definition that all possible poles of $f(u,v,z)$ on the u plane are limited to the lattice points $u = m/2 + n\tau/2$ and $u = -v + m + n\tau$ ($m, n \in \mathbb{Z}$). In view of the quasi-periodicity, therefore, we have only to verify that $f(u,v,z)$ is non-singular at $u = 0, 1/2, \tau/2, 1/2 + \tau/2$, and $-v$.

The absence of poles at $u = 0, 1/2$ and $-v$ can be verified by straightforward calculations on the basis of the singular behavior of $x(u,z)$, $x^{(1/2)}(u,z)$ and $x^{(2)}(u,z)$ as $u \to 0$.

In order to examine the points $u = \tau/2$ and $u = 1/2 + \tau/2$, one has to examine the singular behavior of $x(2u,z)$ and $y(2u,z)$ as $u \to \tau/2, 1/2 + \tau/2$. This can be worked out by combining the quasi-periodicity of $x(u,z)$ and $y(u,z)$ and their singular behavior as $u \to 0$:

1. As $u \to \tau/2$,

$$\begin{aligned} x(2u,z) &= e^{2\pi iz} x(2u-\tau,z) = e^{2\pi iz}\left(\frac{1}{2u-\tau} + O(1)\right), \\ y(2u,z) &= 2^{2\pi iz} y(2u-\tau,z) = e^{2\pi iz}\left(-\frac{1}{(2u-\tau)^2} + O(1)\right). \end{aligned} \tag{C.7}$$

2. As $u \to 1/2 + \tau/2$,

$$\begin{aligned} x(2u,z) &= e^{2\pi iz} x(2u-1-\tau,z) = e^{2\pi iz}\left(\frac{1}{2u-1-\tau} + O(1)\right), \\ y(2u,z) &= e^{2\pi iz} y(2u-1-\tau,z) = e^{2\pi iz}\left(-\frac{1}{(2u-1-\tau)^2} + O(1)\right). \end{aligned} \tag{C.8}$$

Using these observations, one can confirm the absence of poles of $f(u,v,z)$ at $u = \tau/2$ and $1/2 + \tau/2$ by direct calculations.

We can thus verify that $f(u,v,z)$ is indeed an entire function on the u plane.

C.3 Proof of (B.21) – (B.26)

Rather than directly proving these identities, let us prove them in a differentiated form. For illustration, we consider the first identity (B.21). Differentiating this identity by u gives

$$\begin{aligned}&x(u,2z)y^{(1/2)}(-u,2z)-y(u,2z)z^{(1/2)}(-u,2z)+x^{(1/2)}(u,2z)y(-u,2z)\\&-y^{(1/2)}(u,2z)x(-u,2z)=2\wp'(u).\end{aligned} \tag{C.9}$$

One can prove it directly, repeating the complex analytic reasoning that we have presented in other cases. An alternative way is to take the limit, as $v\to u$, of the functional identity

$$\begin{aligned}&x(2u,2z)y^{(1/2)}(-u-v,2z)-y(2u,2z)x^{(1/2)}(-u-v,2z)+x^{(1/2)}(u+v,2z)y(-2v,2z)\\&-y^{(1/2)}(u+v,2z)x(-2v,2z)-x(u-v,z)(\wp(2u)-\wp(2v))=0.\end{aligned} \tag{C.10}$$

(This yields the above identity upon substituting $u\to u/2$ and $v\to v/2$.) This functional identity can be derived by the same method as the proof of (B.35) – (B.38).

Similarly, the third and fifth of (B.21) – (B.26) are obtained from the following functional identities:

$$\begin{aligned}&2x(u,2z)y^{(2)}(-u-v,z)-y(u,2z)x^{(2)}(-u-v,z)+x^{(2)}(u+v,z)y(-2v,2z)\\&-2y^{(2)}(u+v,z)x(-2v,2z)-x(u-v,z)(\wp(u)-\wp(v))=0,\end{aligned} \tag{C.11}$$

$$\begin{aligned}&2x^{(1/2)}(u,2z)y^{(2)}(-u-v,z)-y^{(1/2)}(u,2z)x^{(2)}(-u-v,z)+x^{(2)}(u+v,z)y^{(1/2)}(-2v,2z)\\&-2y^{(2)}(u+v,z)x^{(1/2)}(-2v,z)-x(u-v,z)(\wp(u)-\wp(v))=0.\end{aligned} \tag{C.12}$$

The second, forth and sixth of (B.21) – (B.26) can be similarly derived from the last three of (B.35) – (B.38). This completes the proof of the functional identities.

We conclude this appendix with a comment on the "const." terms of these identities. In principle, these terms can be determined by examining the identities at a special point of the u plane. Let us consider, e.g., (B.21). At $u=z$, the first term on the left hand side vanishes. Evaluating the other terms at this point, therefore, one finds that

$$\text{const.}=2\wp(z)-x^{(1/2)}(z,2z)x(-z,2z). \tag{C.13}$$

The same formula can be reproduced by substituting $u=-z$. One can similarly derive an explicit expression for the other identities.

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